MATH 2301 03 Sep 2020

* Recap: The element $\zeta \in \mathcal{A}_P$ is invertible for any (finite) poset P. Its inverse is called μ, the Möbius function.

- We derived a recursive formula for μ:

$$\mu([x,x]) = 1$$
$$\mu([x,y]) = -\sum_{x \le z < y} \mu([x,z])$$

- For divisor poset of some n, we also have

$$\mu([1,n]) = \begin{cases} 0 & \text{if } n \text{ not squarefree} \\ (-1)^k & \text{where } k = \#\text{ of distinct prime factors of } n = \omega(n) = \text{omega}(n) \end{cases}$$

* Today: Möbius inversion + one-sided convolution

Thm: (Möbius inversion) Let $f, g \in \mathcal{A}_P$.

① If $g = (f * \zeta)$, then $(g * \mu) = f$

② If $g = (\zeta * f)$, then $(\mu * g) = f$.

Pf: Follows from $\zeta * \mu = \delta = \mu * \zeta$ + associativity of $*$

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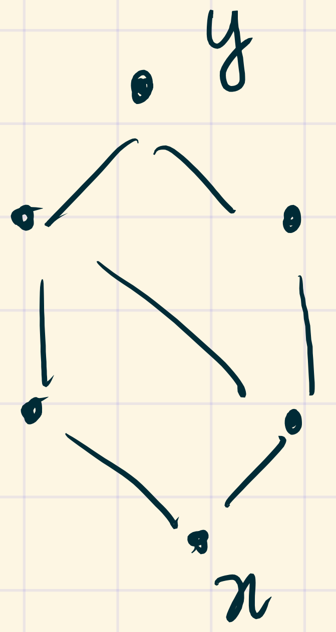
* One-sided convolution. Let p, q be functions on P, i.e. $p, q : P \to \mathbb{R}$ ← often these come up naturally.



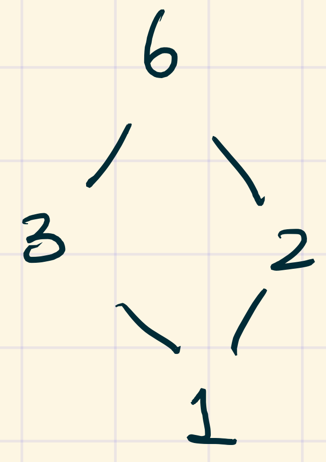
Let $f \in \mathcal{A}_P$. We can define the one-sided convolutions $f * p$, $q * f : P \to \mathbb{R}$

$$(f*p)(x) = \sum_{x \le z} f([x,z]) \cdot p(z)$$

$$(q*f)(y) = \sum_{z \le y} q(z) \, f([z,y])$$

Under matrix representation, you can reinterpret this as multiplying a matrix with a vector.

E.g. Divisor poset of 6 :



Let $p : P \to \mathbb{R}$ be the identity function $\boxed{p(x) = x.}$

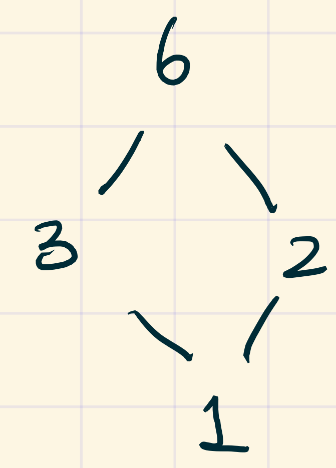
Consider $\boxed{\zeta \in A_P.}$

$$(\zeta * p)(x) = \sum_{x \leq z} \zeta([x, z]) \cdot p(z)$$

$$(\zeta * p)(x) = \sum_{x \leq z} z \quad ; \quad (\zeta * p)(1) = 12 ; \ (\zeta * p)(2) = 8$$

$$(\zeta * p)(3) = 9 \ ; \ (\zeta * p)(6) = 6$$

$$(p * \zeta)(x) = \sum_{z \leq x} p(z) \cdot \zeta([z, x])$$



$$(p * \zeta)(x) = \sum_{z \leq x} z.$$

$(p * \zeta)(1) = 1$, $(p * \zeta)(2) = 3$, $(p * \zeta)(3) = 4$

$(p * \zeta)(6) = 12.$

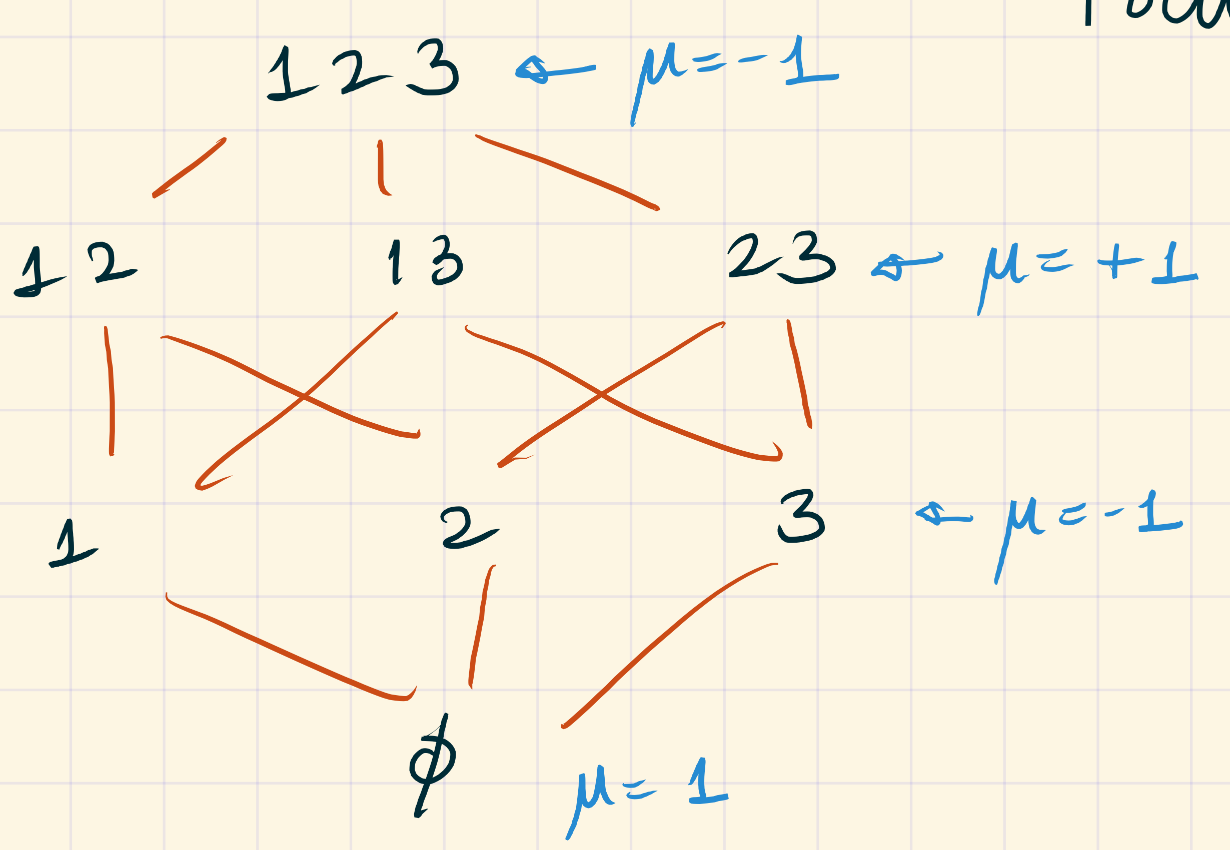
Thm (Möbius inversion, one-sided) : Let $p, q : P \to \mathbb{R}$

① If $q = \zeta * p$, then $p = \mu * q$.

② If $q = p * \zeta$, then $p = q * \mu$.

Pf : Similar to previous + show that 2-sided & one-sided convolution is "compatible"

E.g. Subset poset



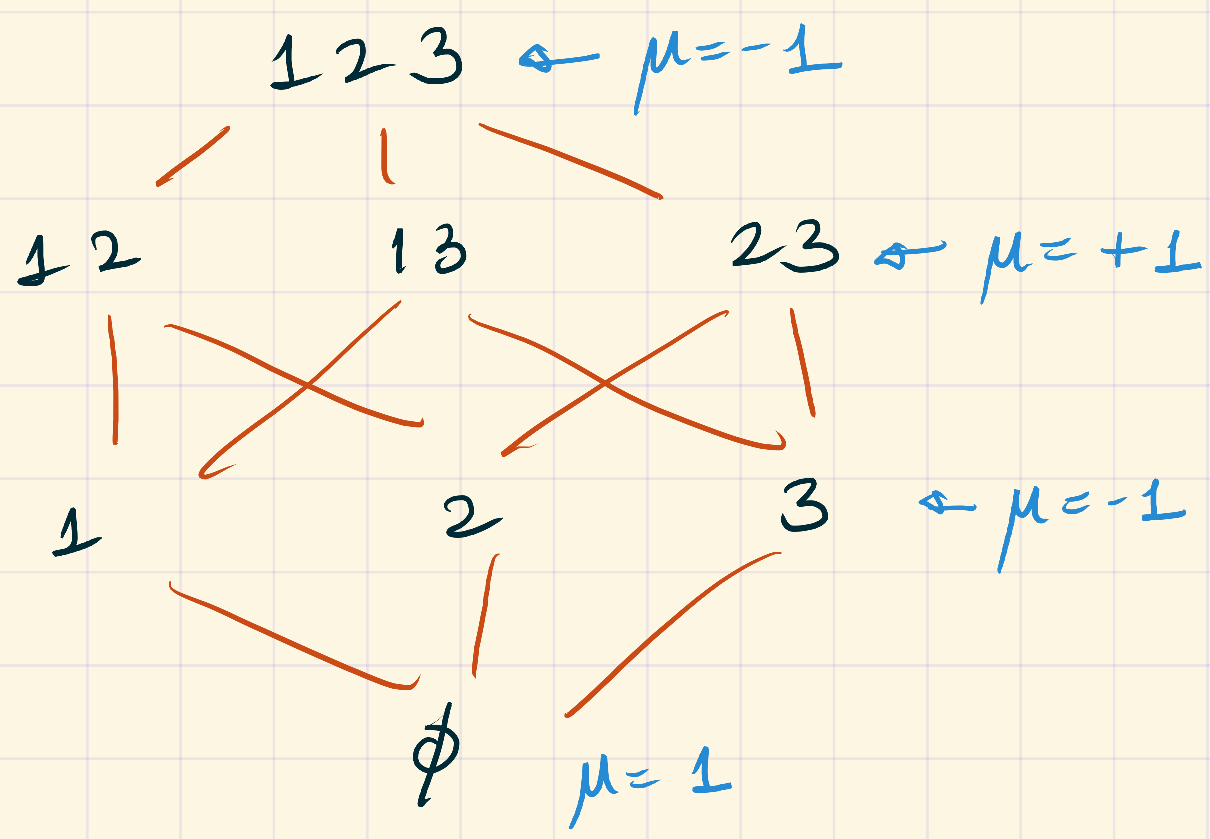
Focus on $\mu([\emptyset, A])$

Thm: Let S be a finite set & $(P, \leq)$ be the subset poset of S.

Then

① $\mu([\emptyset, A]) = (-1)^{|A|}$

② $\mu([B, A]) = (-1)^{|A \setminus B|}$



Pf: Let S a finite set.

Let $A \subseteq S$, of size n.

Use induction:

Base case: $\mu([\phi, \phi]) = 1$

Induction hypothesis: for any B such that $|B| < n$, we have

$\mu([\phi, B]) = (-1)^{|B|}$

Recursive formula for $\mu([\phi, A])$

$$\mu([\phi, A]) = -\sum_{\phi \subseteq B \subsetneq A} \underbrace{\mu([\phi, B])}_{\text{known by induction.}}$$

$$= -\sum_{k=0}^{n-1} \sum_{\substack{\phi \subseteq B \subseteq A \\ |B| = k}} \mu([\phi, B])$$

$$= -\sum_{k=0}^{n-1} \underbrace{\sum_{\substack{|B| = k \\ B \subseteq A}}}_{\text{has } \binom{n}{k} \text{ terms}} (-1)^k .$$

$$\mu([\phi, A]) = -\boxed{\sum_{k=0}^{n-1} \binom{n}{k} (-1)^k}$$

Note: by the binomial theorem, we know:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^n = 0$$

$$\binom{n}{n} (-1)^n + \boxed{\sum_{k=0}^{n-1} \binom{n}{k} (-1)^k} = 0$$

$$\Rightarrow \quad -\sum_{k=0}^{n-1} \binom{n}{k} (-1)^k = \binom{n}{n} (-1)^n = (-1)^n$$

$$\Rightarrow \mu([\phi, A]) = (-1)^n \quad \text{if} \quad n = |A|.$$

* The inclusion-exclusion principle:

Suppose S is a set of "properties" that objects may / may not satisfy.
$S = \{1, 2, \ldots n\}$

You want to count the number of objects satisfying certain properties.

$(P, \leq)$ = subset poset of S.

Define two functions $f, g : P \to \mathbb{R}$

① $f(A)$ = number of objects satisfying <u>exactly</u> the properties in A, and none of the properties outside A

② $g(A)$ = number of objects satisfying <u>at least</u> the properties in A.

<u>Note</u>: $g = (\zeta * f)$ because:

$$g(A) = \sum_{A \subseteq B} f(B)$$