MATH 2301 04 Sep 2020

Congratulations on finishing the midterm exam!

* Recap: ① μ for the subset poset of some set S

$$\mu([A,B]) = (-1)^{|B\setminus A|} \quad \text{whenever } A \subseteq B$$

$B\setminus A = \{b \in B \mid b \notin A\}$ = set difference

[using clever counting & binomial expansion of $(1-1)^n$.]

② If $p : P \to \mathbb{R}$ & $f \in \mathcal{A}_P$,

One-sided convolutions (for general posets)

$(f*p) : P \to \mathbb{R}$ is defined as

$$(f*p)(x) = \sum_{x \preceq z} f([x,z])\, p(z)$$

$(p*f) : P \to \mathbb{R}$ is defined as

$$(p*f)(y) = \sum_{z \preceq y} p(z)\, f([z,y])$$

(Möbius inversion for one-sided convolution)

③ If $q = p*\zeta$ then $q*\mu = p$

If $q = \zeta * p$ then $\mu * q = p$

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* Inclusion-exclusion principle

Used to count number of objects satisfying / not satisfying certain collections of properties.

$S = \{1, \ldots, n\}$ ← represents a set of properties that may / may not be satisfied.

E.g. $S = \{1, 2, 3\}$

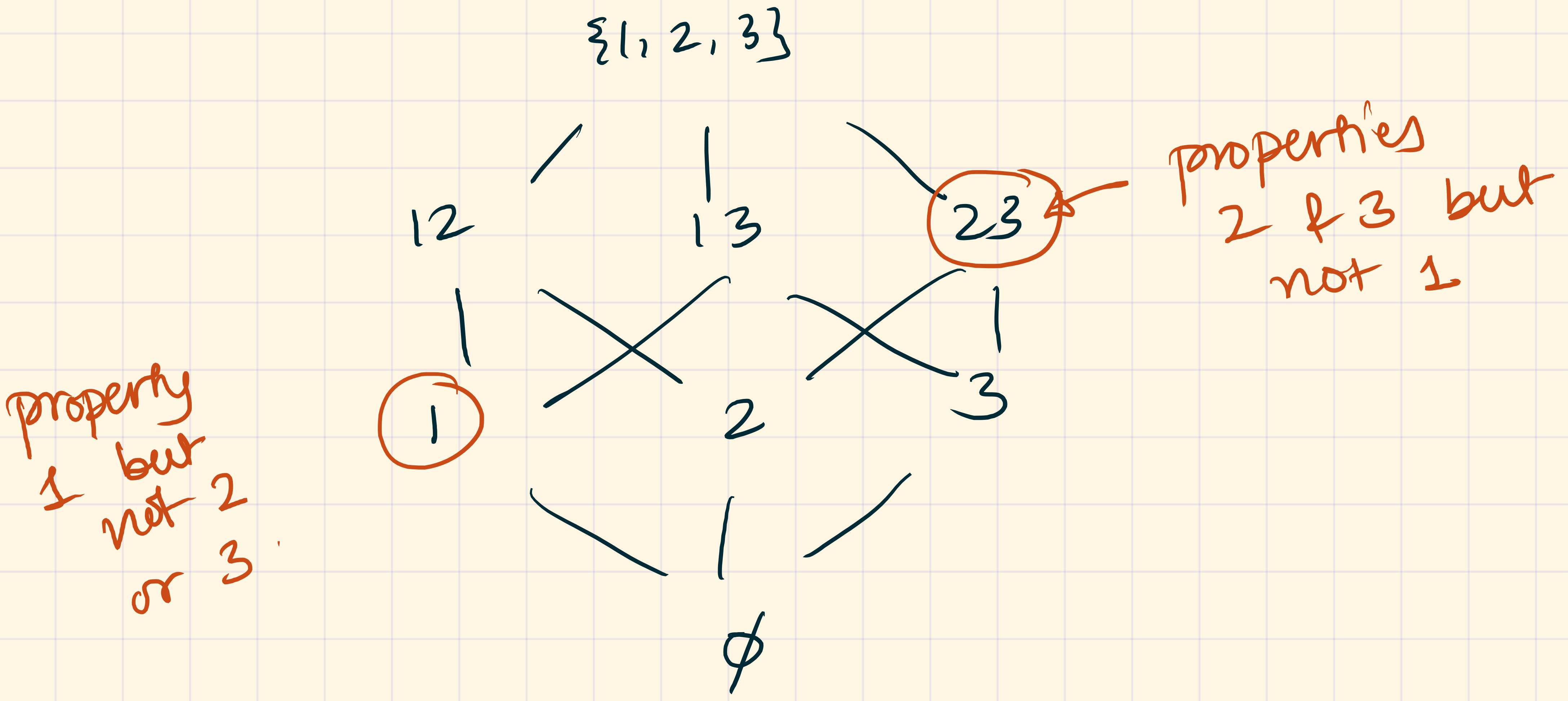
ordered dessert

ordered main

ordered drink

(when some people go to a restaurant)

We use the subset poset of S:



$(P, \preceq)$

Let's define two functions on P:

1. $p : P \to \mathbb{R}$

$p(A) =$ number of objects satisfying $\boxed{\text{at least}}$ the properties in A (and possibly other properties)

E.g. In the worksheet, we had:

$p(\{1\}) = 5$ ← 5 people ordered dessert (and possibly other stuff)

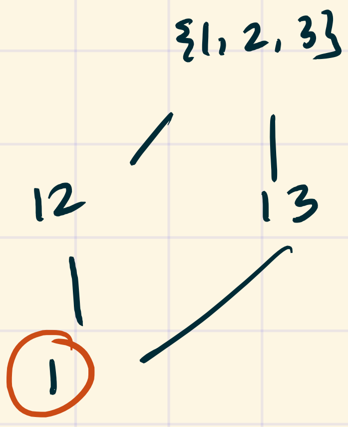
$p(\{1, 3\}) = 3$ ← 3 ordered drink + dessert (and possibly other stuff)

2. $q : P \to \mathbb{R}$

$q(A) =$ number of objects satisfying $\boxed{\text{exactly}}$ the properties in A, and no other properties.

In worksheet, this was not made explicit

From this, we see that:



$A = \{1\}$

$B =$ anything above it

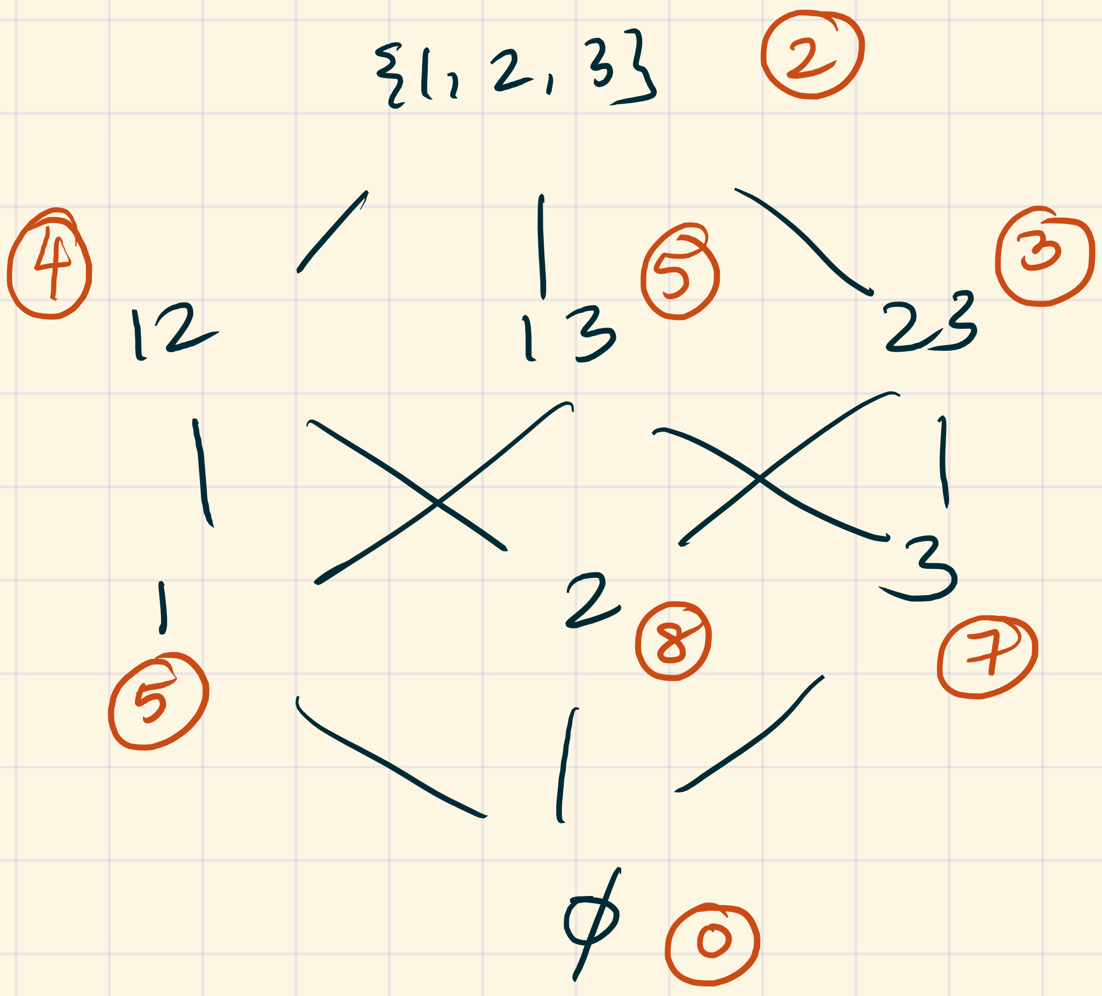
$$p(A) = \sum_{A \subseteq B}' q(B)$$

$$= p(A) = \sum_{A \subseteq B}' \zeta([A, B])\, q(B)$$

$$\Rightarrow \boxed{p = \zeta * q} \Longrightarrow \text{(Möbius inversion)} \quad \boxed{q = \mu * p}$$

"at least" "exactly"

p-values (from worksheet)



Q: How many people ordered drink + dessert but nothing else?

i.e what is $q(\{1,3\})$?

$$q(\{1,3\}) = (\mu * p)(\{1,3\})$$

$$= \sum_{\{1,3\} \subseteq B} \underbrace{\mu([\{1,3\}, B])}_{= (-1)^{|B \setminus \{1,3\}|}} \cdot p(B)$$

$$q(\{1,3\}) = (-1)^{|\{1,3\} \setminus \{1,3\}|} \cdot p(\{1,3\}) + (-1)^{|\{1,2,3\} \setminus \{1,3\}|} p(\{1,2,3\})$$

$$= (-1)^0 5 + (-1)^1 2 = 5 - 2 = 3.$$

Include everything satisfying 13 & exclude everything satisfying 123

Let's do another one:

$$q(\{2\}) = (\mu * p)(\{2\})$$
$$= \sum_{\{2\} \subseteq B}' \mu([\{2\}, B]) \cdot p(B)$$

B has 4 options: $\{2\}$, $\{1,2\}$, $\{2,3\}$, $\{1,2,3\}$

$$(-1)^0 p(\{2\}) + (-1)^1 p(\{1,2\}) + (-1)^1 p(\{2,3\}) + (-1)^2 p(\{1,2,3\})$$

include   exclude   include

$$= 8 - 4 - 3 + 2 = 3$$

← exactly 3 people ordered main, but nothing else.

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* More interesting example : counting "derangements"

Defn: A derangement is a permutation of n objects in which no object is fixed.

E.g. 3 objects 1, 2, 3:

~~1 2 3~~

"fixed"

~~1 3 2~~

~~2 1 3~~

2 3 1 ✓

3 1 2 ✓

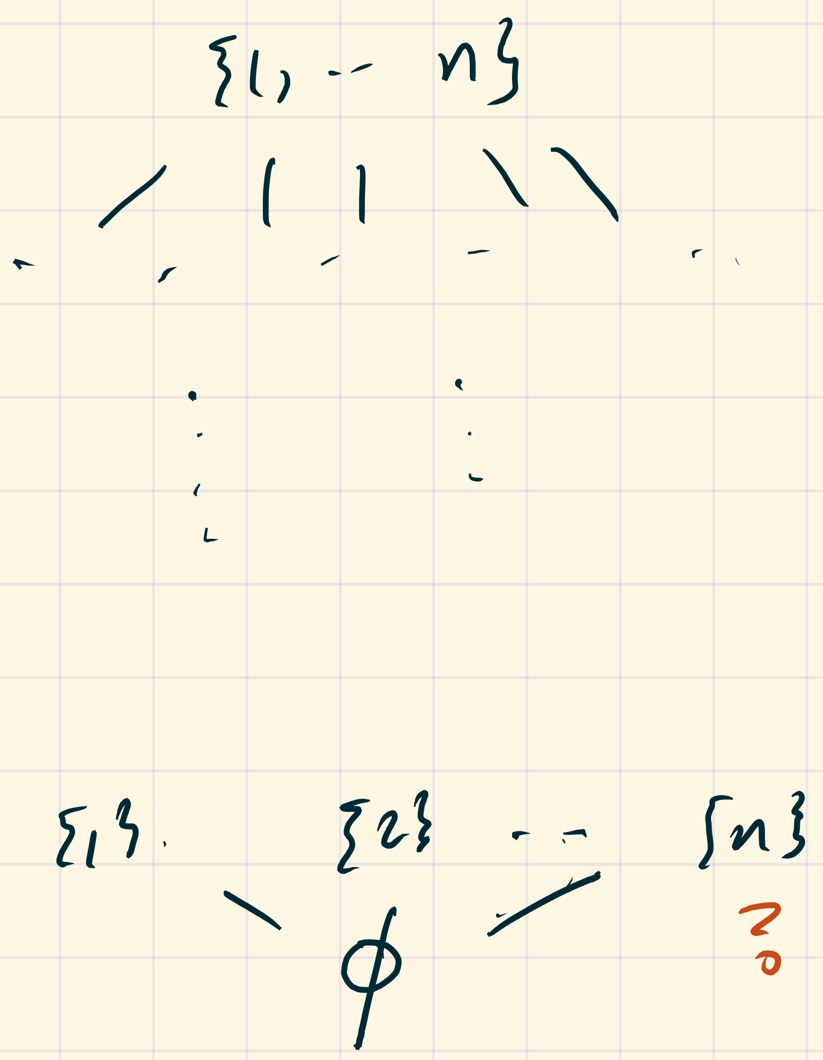
~~3 2 1~~

derangements

Q: How many derangements are there of n elements?

(There are $n!$ permutations ...)

Consider $S = \{1, \dots, n\}$: represents n properties, where property i says that the i^{th} element is fixed.



$p : P \to \mathbb{R}$

$p(A) = \#$ permutations fixing at least the elements of A

$q(A) = \#$ permutations fixing exactly the elements of A

$? = \#$ derangements $= q(\phi)$.

$n! = \#$ permutations $= p(\phi)$

Note: $p = \zeta * q \Rightarrow q = \mu * p$

$$q(\phi) = (\mu * p)(\phi) = \sum_{\phi \subseteq B} \mu([\phi, B]) \cdot p(B)$$

much easier than q!

5 elements:

fix 1 & 2 $\Rightarrow$ count $p(\{1,2\}) = 3!$

1 2 _ _ _

$p(B) = (n-k)!$
where $k = |B|$.