MATH 2301 23 Sept 2020

Welcome back everyone!

* Recap : (From a long time ago):

Möbius inversion on posets / inclusion-exclusion

$S = \{1, 2, \ldots, n\}$ a set of properties

$(P, \preceq)$: subset poset of S ; let $A \subseteq S$

$p(A)$ = # objects satisfying at least properties of A

$q(A)$ = # objects satisfying exactly properties of A

$$p(A) = \sum_{A \subseteq B} q(B) = (\zeta * q)(A)$$

Note: typo fixed in this formula

$$\Rightarrow q(A) = (\mu * p)(A) = \sum_{A \subseteq B} \mu([A,B])\, p(B)$$

in subset poset

$$\boxed{q(A) = \sum_{A \subseteq B} (-1)^{|B \setminus A|} p(A)}$$

* Often for us, we'll want to compute $q(\emptyset)$.

* Example : Counting derangements of n objects

= # permutations where each object moves

$S = \{1, 2, \ldots, n\}$; i represents the property that the i^{th} object is fixed

If $A \subseteq S$,

$p(A)$ = # permutations in which at least everything in A is fixed

$q(A)$ = # permutations in which exactly the things in A fixed.

$*$ Number of derangements $= g(\phi)$

$$g(\phi) = \sum_{\phi \subseteq B} (-1)^{|B \setminus \phi|} \; p(B)$$

$$g(\phi) = \sum_{B} (-1)^{|B|} p(B)$$

$*$ Note: $p(B) = (n - |B|)!$ [Just ignore everything in B & move the rest]

$$g(\phi) = \sum_{k=0}^{n} \sum_{|B|=k} (-1)^{|B|} \; p(B)$$

$$= \sum_{k=0}^{n} \sum_{|B|=k} \boxed{(-1)^{k} \; (n-k)!}$$ ← appears $\binom{n}{k}$ times

$\exists \binom{n}{k}$ subsets B that have size k.

$$g(\phi) = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \cdot (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!\,(n-k)!} \; (n-k)!$$

$$g(\phi) = \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!} = n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots$$

E.g. # derangements on 3 letters

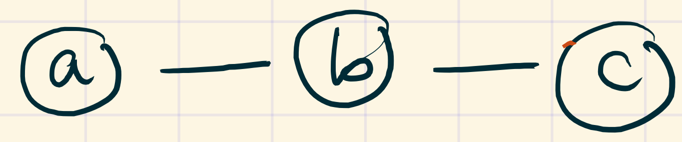
$$3! - \frac{3!}{1!} + \frac{3!}{2!} - \frac{3!}{3!} = 6 - 6 + 3 - 1 = 2.$$

If $n=4$: $4! \left[1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right] = \frac{4!}{24} \left[24 - 24 + 12 - 4 + 1 \right]$

$= 9$ ⇝ $\exists$ 9 derangements on 4 letters.

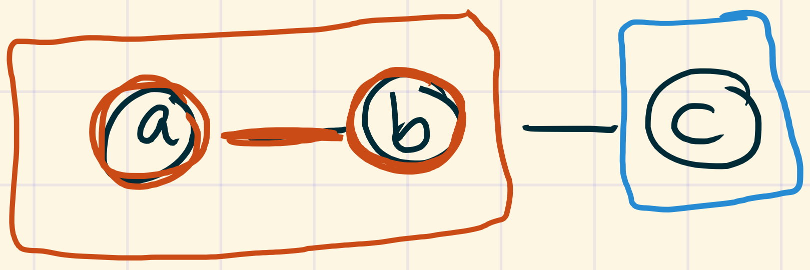
Möbius inversion & graph colouring.

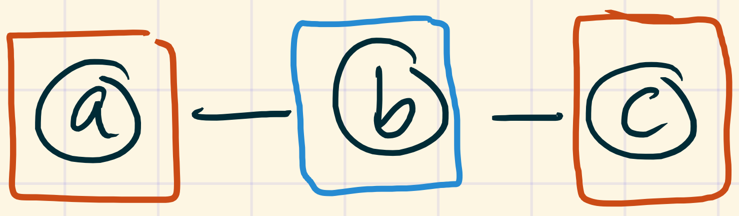
(undirected w/o self loops)

Let G be a graph (V, E): 

<u>Defn</u>: A <u>bond</u> of G is a <u>partition</u> of V $B = \{V_1, V_2, \dots, V_k\}$ where $V = \coprod_{i=1}^{k} V_i$ such that for every i, the subgraph of G whose edges all start & end in V_i, is connected, and $V_i \neq \emptyset$.

[$\sqcup = \amalg$ means disjoint union, i.e. union where the sets are disjoint, or non-overlapping.]

<u>E.g.</u> • $B = \{\{a,b\}, \{c\}\}$ is a bond. 

$B = \{\{a,c\}, \{b\}\}$ is not a bond. 

because all paths connecting a to c pass through b, which is not in $\{a,c\}$.

are bonds.

$B = \{\{b,c\}, \{a\}\} = bc/a$ (shorthand notation)

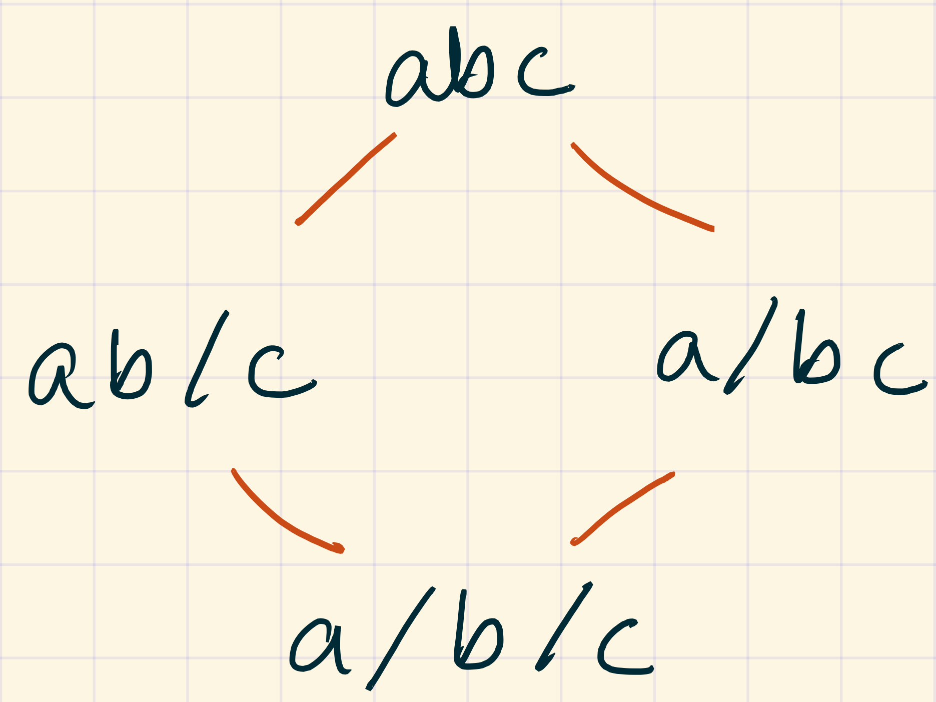
$B = a/b/c = \{\{a\}, \{b\}, \{c\}\}$.

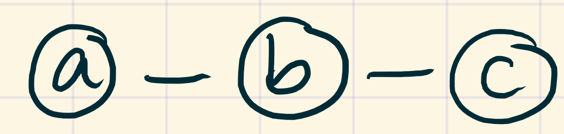
$B = abc = \{\{a,b,c\}\}$

* <u>Bond poset</u>: The set of bonds of G forms a poset, as follows. Let $B = \{V_1, \dots V_k\}$ & $C = \{W_1, \dots, W_\ell\}$

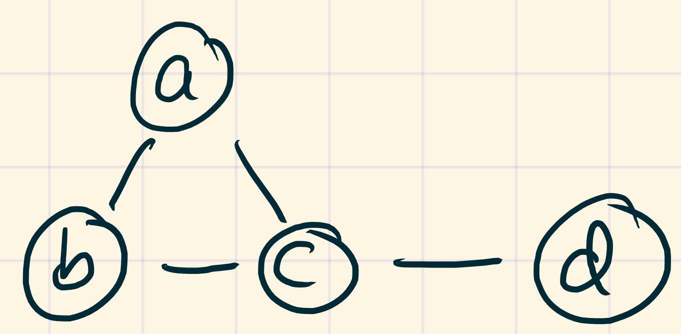
Then $B \preceq C$ if $\forall\ 1 \leq i \leq k$, $\exists\ 1 \leq j \leq \ell$ such that $V_i \subseteq W_j$

E.g. :





E.g.



Bonds :

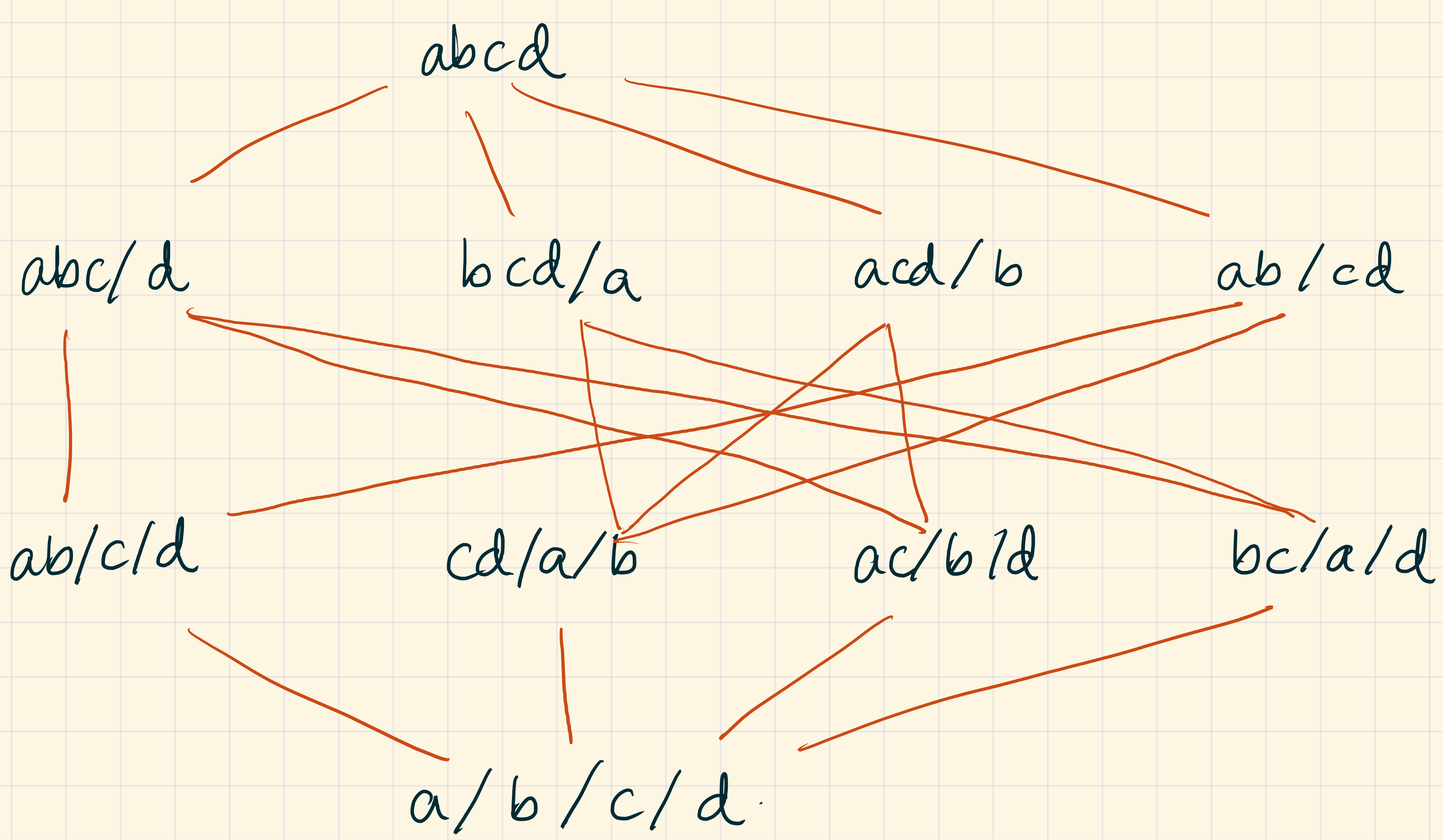
1 piece : abcd

2 pieces : abc/d , ~~abd/c~~ , acd/b , bcd/a
ab/cd

3 pieces : ab/c/d , cd/a/b , ac/b/d , bc/a/d.

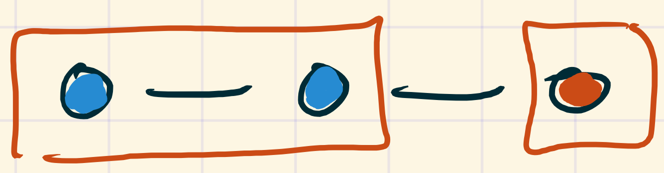
4 pieces : a/b/c/d.

Hasse diagram :



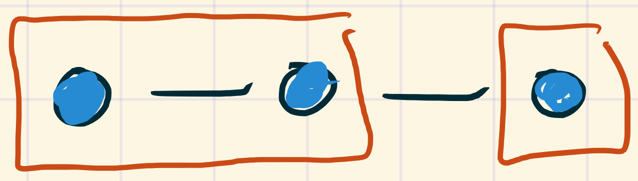
<u>Defn</u>: Let $G=(V,E)$ be a graph. Let $B=\{V_1, \dots, V_\ell\}$ be a bond. A k-colouring of the bond B is an assignment of one of k colours to each vertex, such that:

* If $v, w \in V_i$ they get the same colour.



$\rightsquigarrow$ a proper 2-colouring of this bond.

Bond partition



$\rightsquigarrow$ an improper 2-colouring (or 1-colouring) of this bond.

* A k-colouring of a bond is <u>proper</u> if: whenever V_i & V_j are connected by an edge, they get different colours.

Otherwise, it is an improper colouring.