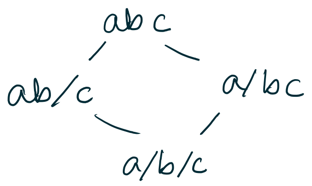
MATH 2301 24 Sept 2020

* **Recap**: Given a graph (V,E) [undirected w/o self loops] we can write down its bond poset:

A bond $B = \{V_1, \dots, V_k\}$ is $\leq$ a bond $C = \{W_1, \dots, W_\ell\}$ if every V_i is entirely contained within a single W_j. Equivalently, each W_j is a union of some number of pieces V_i.

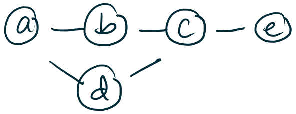
* **Example**: ⓐ — ⓑ — ⓒ



* **Rmk**: In fact, the bond poset is a lattice.

Exercise: Figure out the lub/glb in this poset

E.g.

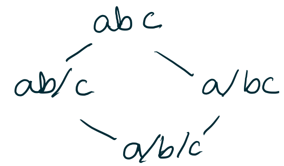


(ad/b/ce) & (a/b/cd/e)

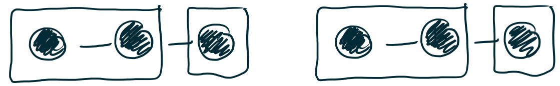
glb is a/b/c/d/e

lub? ad & cd must be clubbed & ce is also clubbed
= acde/b.

E.g. ⓐ — ⓑ — ⓒ



Recall: A k-colouring of a bond $B = \{V_1, \dots, V_\ell\}$ means that you assign one of k different colours to each V_i, such that all vertices in V_i are coloured with that colour.



proper 2-colouring

2-colourings.

A proper k-colouring is one where if V_i & V_j are connected by an edge, they get different colours.

* **Notation**: Let B_0 be the smallest bond
$$B_0 = \{\{v\} \mid v \in V\}.$$

Note: A proper k-colouring of B_0 is exactly a proper k-colouring of the graph G.

Defn: Let $p_t(B)$ = number of t-colourings of a bond B

$q_t(B)$ = number of proper t-colourings of a bond B.

We want to compute $q_t(B_0)$ = # of proper t-colourings of the graph G.

Note: $p_t(B) = t^{\#\text{parts in } B}$

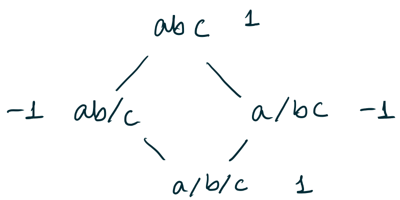
And: $p_t(B) = \sum_{B \leq C} q_t(C)$. Convince yourself!

$p_t(B) = (\zeta * q_t)(B)$.

Möbius inversion $\Rightarrow$ $q_t(B) = (\mu * p_t)(B)$.

$$q_t(B_0) = \sum_{C} \mu([B_0, C]) \cdot p_t(C).$$

Example :



$\mu([B_0, -])$

$$q_t(B_0) = 1 \cdot t^3 - 1 \cdot t^2 - 1 \cdot t^2 + 1 \cdot t$$
$$= t^3 - 2t^2 + t = t(t-1)^2$$

Theorem : The chromatic polynomial of G has the following formula :

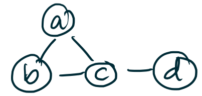
$$P_G(t) = \sum_{C \text{ bond}} \mu([B_0, C]) \cdot p_t(B)$$
$$= \sum_{C \text{ bond}} \mu([B_0, C]) \cdot t^{(\#\text{ parts of } B)}.$$

* Rmk : This method also proves that $P_G(t)$ is a polynomial!

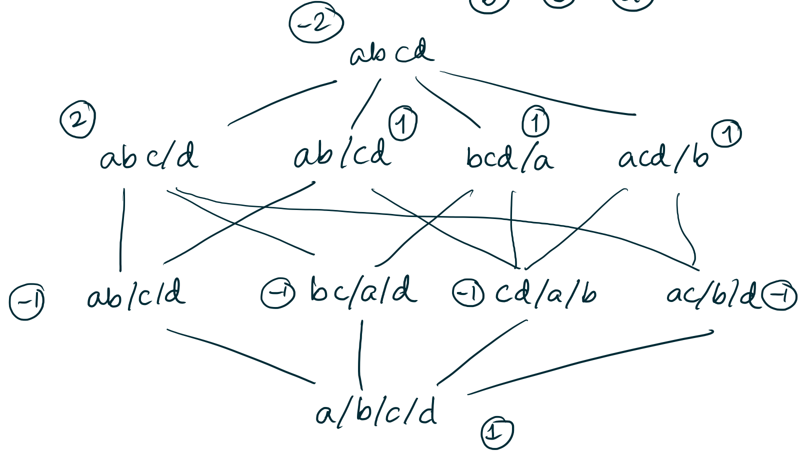
Recall : $\mu([x,x]) = 1 \quad \forall x$.

$$\mu([x,y]) = -\sum_{x \le z < y} \mu([x,z])$$

Example (Last time) :



$\mu([B_0, -])$



$$P_G(t) = t^4 - 4t^3 + 5t^2 - 2t$$
$$= t(t-1)^2(t-2) \quad \rightsquigarrow \text{ can check directly in this case}$$

Exercise : Do this for the graph in the worksheet!

* Done with posets for this course

"Machines" — Regular expressions & finite automata.

A way of pattern-matching, for example for English words.

Defn : An alphabet is a finite set of symbols; typically denoted Σ. (sigma)

E.g. $\Sigma = \{a, b, c\}$ or $\Sigma = \{0, 1\}$

$\Sigma = \{a, b, c, \cdots, z\}$

Defn : A word in an alphabet Σ is a finite ordered list of elements in Σ, written without punctuation next to each other.

Eg. if $\Sigma = \{0,1\}$ then 000, 111, 101011 are words

The "empty word" is also a word; it is denoted ε.
(So as a convention, Σ should not contain the symbol ε)

The set of all possible words of Σ, including the empty word, is denoted Σ^*.
(Σ^* is infinite unless $\Sigma = \phi$; in that case, $\Sigma^* = \{\varepsilon\}$)

Defn : A language L is just a subset of Σ^*.
(L tells you which words are valid words in that language)