

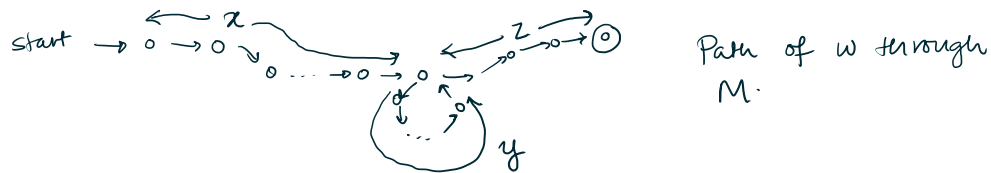
* We're investigating properties of regular languages.

(1) Let L be a regular language; let M be a DFA that recognises L . Let $n =$ number of states in M .

aside } (2) Suppose that L is infinite
 \Rightarrow There must be a word in L that has length $> n$.

Let $w \in L$ such that $|w| > n$.

$\Rightarrow w$ repeats a state in M (being longer than the # of states)



We know that

$w = xyz$ such that

- (1) $|y| \geq 1$, $|x| \geq 0$, $|z| \geq 0$, and
- (2) y begins and ends at the same state

* Conclusion: All of the following words are also in L :

xz , xyz , $xyyz$, $xyyyz$, ..., xy^mz for any $m \geq 0$.

* Note: There may be other ways to break up w so that the middle portion repeats states.

The main point is that there is at least one possibility.

* We'll use this observation to identify non-regular languages

* The pumping lemma:

Let L be a regular language. Then there is some number n (called "the pumping length") such that if $w \in L$ & $|w| > n$, then we can write $w = xyz$ with the following properties:

- (1) $|y| \geq 1$
- (2) $|xy| < n$
- (3) For each $m \geq 0$, the string $xy^mz \in L$.

* The argument on the previous page is a proof sketch.

* Now let's use this:

E.g. $L = \{0^k 1^k \mid k \geq 0\}$. Let's prove that L is not regular.

We prove by contradiction: let's assume that L is regular.

Then L satisfies the conditions of the pumping lemma; i.e. there is some pumping length n beyond which the three conditions hold.

* Suppose that we have some $w \in L$, such that $|w| > n$, for example, consider $0^n 1^n \in L$.

* We'll show that this string cannot satisfy the conditions of the pumping lemma!

Let's try to divide up $0^n 1^n$ into xyz

satisfying the 3 conditions.

(2) $|xy| < n \Rightarrow xy$ consists only of 0s.

(1) $|y| \geq 1 \Rightarrow y$ is of the form 0^k for $k \geq 1$.

(3) $xy^m z \in L$ for all $m \geq 0$

$\Rightarrow xz, xyz, xyyz, xy^2yz, \dots$ are all in L .

E.g. $n=4$

00001111
 $\underbrace{\hspace{1.5cm}}_x \underbrace{\hspace{1.5cm}}_y \underbrace{\hspace{1.5cm}}_z$

$xyyz:$

000001111

$xy^2yz:$

0000001111

} can't be in L because they don't satisfy the conditions of L !

(1) $\Rightarrow y = 0^k$ for some $k \geq 1$

(2) $\Rightarrow x = 0^l$ for some $l \geq 0$

$w = \underbrace{0^l}_x \underbrace{0^k}_y \underbrace{0^{n-k-l} 1^n}_z$

Pump this string, let's say to

$xyyz:$

$\underbrace{0^l}_x \underbrace{0^k}_y \underbrace{0^k}_y \underbrace{0^{n-k-l} 1^n}_z$

$= 0^{n+k} 1^n = xy^2yz \in L$

This is a contradiction!

To conclude, we found a string $w \in L$, with $|w| > n$, that cannot be pumped. This means that L can't be regular!

E.g. $L = \{w \mid w \text{ has an equal number of 0s \& 1s}\}$

Let's prove that L is not regular.

① For contradiction, assume that it is regular.

Let $n =$ pumping length.

② Let's find a string $w \in L$, with $|w| > n$, that can't be pumped.

$w = 0^n 1^n$ still works!

If $w = xyz$ satisfying the conditions 1 & 2, then

$|xy| < n \Rightarrow xy$ consists of 0s

$|y| \geq 1 \Rightarrow y$ consists of at least one 0.

Then $xy^m z$ ($m > 1$) has too many 0s!

(The extra copies of y introduce many 0s, but you only still have n 1s, in z)

E.g. $\Sigma^* = \{a\}$

$L = \{a^{k^2} \mid k > 0\}$

$L = \{a, aaaa, aaaaaaaaa, a^{25}, a^{36}, \dots\}$

Claim: L is not regular.

1) Assume for contradiction that L is regular. $\Rightarrow \exists$ pumping length ($n \geq 2$)

2) Let's try to find a string $w \in L$, with $|w| > n$, that can't be pumped.

Let's try the string $a^{n^2} = w$

$$w = \underbrace{aa \dots a}_x \underbrace{\dots a}_y \underbrace{\dots a}_z$$

$$|xy| < n \Rightarrow |z| > n^2 - n$$

$$|y| > 1$$

Suppose that $x = a^k \quad k \geq 0$

$$y = a^l \quad l \geq 1$$

$$z = a^{n^2 - k - l}$$

What happens if you try to pump this string?

Exercise: Try to derive a contradiction by pumping.