MATH 2301 15 Oct 2020

* We're investigating properties of regular languages.

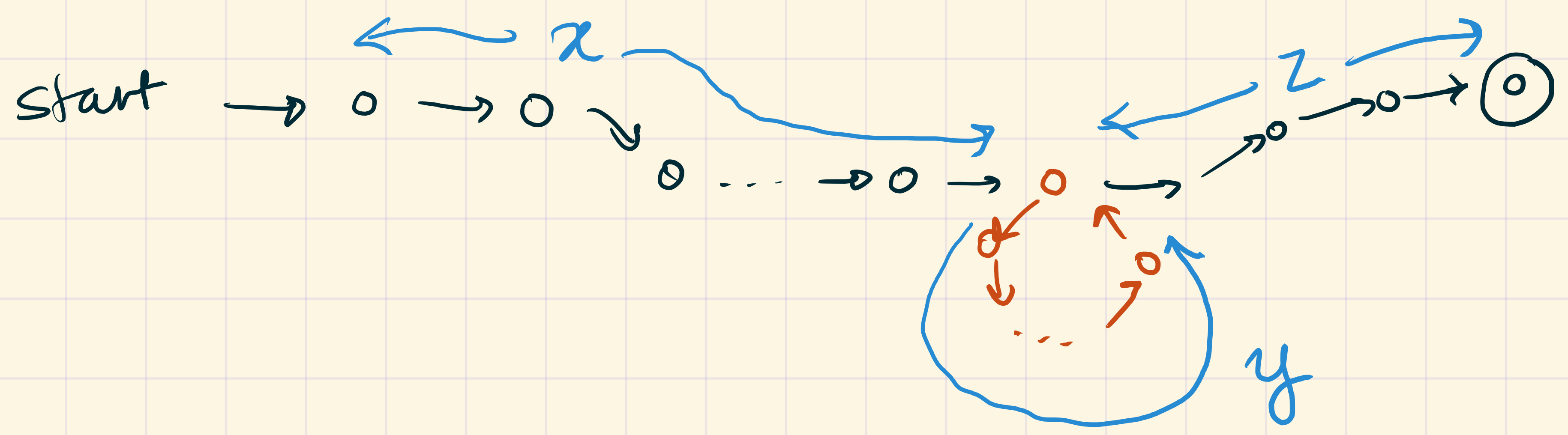
(1) Let L be a regular language; let M be a DFA that recognises L. Let n = number of states in M.

aside (2) Suppose that L is infinite

$\Rightarrow$ There must be a word in L that has length $> n$.

Let $w \in L$ such that $|w| > n$.

$\Rightarrow$ w repeats a state in M. (being longer than the # of states)



Path of w through M.

We know that

$w = xyz$ such that

(1) $|y| \geq 1$, $|x| \geq 0$, $|z| \geq 0$, and

(2) y begins and ends at the same state

* Conclusion: All of the following words are also in L:

xz, xyz, $xyyz$, $xyyyz$, ..., $xy^m z$ for any $m \geq 0$.

* Note: There may be other ways to break up w so that the middle portion repeats states.

The main point is that there is at least one possibility.

* We'll use this observation to identify non-regular languages.

* The pumping lemma:

Let L be a regular language. Then there is some number n (called "the pumping length") such that if $w \in L$ & $|w| > n$, then we can write $w = xyz$ with the following properties:

(1) $|y| \geq 1$

(2) $|xy| < n$

(3) For each $m \geq 0$, the string $xy^m z \in L$.

* The argument on the previous page is a proof sketch.

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* Now let's use this.

E.g. $L = \{0^k 1^k \mid k \geq 0\}$. Let's prove that L is not regular.

We prove by contradiction: let's assume that L is regular.

Then L satisfies the conditions of the pumping lemma; ie there is some pumping length n beyond which the three conditions hold.

* Suppose that we have some $w \in L$, such that $|w| > n$, for example, consider $0^n 1^n \in L$.

* We'll show that this string cannot satisfy the conditions of the pumping lemma!

Let's try to divide up $0^n 1^n$ into xyz satisfying the 3 conditions.

(2) $|xy| < n \Rightarrow xy$ consists only of 0s.

(1) $|y| \geq 1 \Rightarrow y$ is of the form 0^k for $k \geq 1$.

(3) $xy^m z \in L$ for all $m \geq 0$

$\Rightarrow xz, xyz, xyyz, xyyyyz$, etc are all in L.

E.g. $n = 4$

$$\underbrace{00}_{x}\underbrace{\boxed{0}}_{y}\underbrace{0\,1111}_{z}$$

$xyyz$:

00 00 0 1111

$xyyyz$:

00 000 01111

can't be in L because they don't satisfy the conditions of L!

(1) $\Rightarrow y = 0^k$ for some $k \geq 1$

(2) $\Rightarrow x = 0^l$ for some $l \geq 0$

$$w = \underbrace{0^l}_{x}\underbrace{0^k}_{y}\underbrace{0^{n-k-l}\,1^n}_{z}$$

Pump this string, let's say to $xyyz$:

$$\underbrace{0^l}_{x}\underbrace{0^k}_{y}\underbrace{0^k}_{y}\underbrace{0^{n-k-l}\,1^n}_{z}$$

$$= 0^{n+k}\,1^n = xyyz \in L$$

This is a contradiction!

To conclude, we found a string $w \in L$, with $|w| > n$, that <u>cannot</u> be pumped. This means that L can't be regular!

E.g. $L = \{ w \mid w$ has an equal number of 0s & 1s$\}$

Let's prove that L is not regular.

(1) For contradiction, assume that it is regular.

Let n = pumping length.

(2) Let's find a string $w \in L$, with $|w| > n$, that can't be pumped.

$w = 0^n 1^n$ still works!

If $w = xyz$ satisfying the conditions 1 & 2, then

$|xy| < n \Rightarrow xy$ consists of 0s

$|y| \geq 1 \Rightarrow y$ consists of at least one 0.

Then $xy^m z$ $(m > 1)$ has too many 0s!

(The extra copies of y introduce many 0s, but you only still have n 1s, in z.)

E.g. $\Sigma = \{a\}$

$L = \{ a^{k^2} \mid k > 0 \}$

$L = \{ a, aaaa, aaaaaaaaa, a^{25}, a^{36}, \dots \}$

Claim: L is not regular.

1) Assume for contradiction that L is regular. $\Rightarrow \exists$ pumping length $(n \geq 2)$

2) Let's try to find a string $w \in L$, with $|w| > n$, that can't be pumped.

Let's try the string $a^{n^2} = w$

$$w = \underbrace{aa\cdots a}_{x}\,\underbrace{\cdots a}_{y}\,\underbrace{\cdots a}_{z}$$

$|xy| < n \Rightarrow |z| > n^2 - n$

$|y| > 1$

Suppose that $x = a^k$ $\quad k \geq 0$

$y = a^\ell$ $\quad \ell \geq 1$

$z = a^{n^2 - k - \ell}$

What happens if you try to pump this string?

<u>Exercise</u>: Try to derive a contradiction by pumping