MATH 2301 16 Oct 2020

* Pumping lemma:

If L is regular, then there is some $n \in \mathbb{N}$ (pumping length) such that if $w \in L$ & $|w| > n$, then $w = xyz$ such that

① $|xy| < n$ [mostly for convenience]

② $|y| \geq 1$

③ $xy^m z \in L$ for each $m \geq 0$. ["pumping" of w]

E.g. Show that $L = \{a^{k^2} \mid k \geq 1\}$ is not regular.

Pf: Assume for contradiction that L is regular.

Then it has a pumping length n. [we can't choose n]

Let's choose the string (if $n > 1$, otherwise just choose a larger square & the proof will be similar)

$w = a^{n^2}$

we can choose any suitable w as long as $|w| > n$

$|w| > n$ [ignore edge case]

(we can't choose the splitting) $w = xyz$; with $|xy| < n$, $|y| \geq 1$, such that $xy^m z \in L$. for each $m \in \mathbb{N}$.

Let $x = a^k$ for some $k \geq 0$

$y = a^l$ for some $l \geq 1$

$z = a^{n^2-k-l}$

$\left.\right\}$ $\underline{\underline{k+l<n}}$ $\Rightarrow l < n$

Look at the string $xyyz \in L$: $a^k a^l a^l a^{n^2-k-l}$

$= a^{n^2+l} \in L$. Note that $(n+1)^2 = n^2 + 2n + 1$

Since $l < n$, we see that $n^2 < n^2 + l < (n+1)^2$

$\Rightarrow n^2 + l$ cannot be a perfect square. Contradiction!

E.g. $L = \{0^k 1^l 0^k \mid k, l \geq 0\}$. Show that L is not regular

Pf: For contradiction, assume it is regular with pumping length n.

Try: $w = 0^n 0^n \Rightarrow w = xyz$, so $x = 0^p$, $y = 0^q$, $z = 0^{2n-p-q}$

& $p + q < n$.

Pump this: $xy^m z = 0^p 0^{mq} 0^{2n-p-q}$

$= 0^{2n+mq-q} = 0^{2n+(m-1)q} \in L$ if $2n+(m-1)q$ is even.

If q itself is even, which it may be, then the string $xy^m z$ is always in L and there is no contradiction!

So we can't really prove anything with this choice of w!

Instead, let's try $0^n 1 0^n = w$

$w = xyz$, $|xy| < n \Rightarrow x = 0^p$, $y = 0^q$

$z = 0^{n-p-q} 1 0^n$

Now let's try $xyyz$, which is supposed to be in L:

$0^p 0^q 0^q 0^{n-p-q} 1 0^n = 0^{n+q} 1 0^n$.

Since $q > 1 \Rightarrow n+q \neq n$, $xyyz$ doesn't satisfy the condition of being in L. Contradiction!

Games (Impartial combinatorial games)

* We'll have two players
* They make moves in turn according to pre-specified rules.
* The game is perfect information (open game state, no hidden cards etc.)
* We say that a player loses if they can't make any more legal moves for the first time.

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* Subtraction games.

Fix some set of positive integers S (say finite)

Start with a pile of berries.

Each player can eat a number of berries equal to one of the numbers in S.

Whoever runs out of options loses.

E.g. $S = \{1, 3, 4\}$; start with 10 berries

E.g. $S = \{2, 5\}$; start with 18 berries