

* Strategic labelling:

- Draw a graph of all game positions, connecting $A \rightarrow B$ if there is a move from $A \rightarrow B$.
- Label a position as "P" if the previous player wins
- Label a position as "N" if the next player wins (see yesterday's notes)

* The game of nim:

Some number of piles of berries

- You can eat any +ve number of berries from any one pile on your turn.
- The person who eats the last berry wins.

* Analysis:

Easiest case: There is only one pile of berries.

The first player eats everything and wins!

Medium difficulty case: Two piles.

E.g. $(2,3) \rightarrow$ N position (yesterday)

$(2,2) \rightarrow$ P position (yesterday)

E.g.: $(5,7)$ berries

You can use a "mirroring" strategy

\rightarrow Eat two berries out of 7 (player 1) $\leadsto (5,5)$

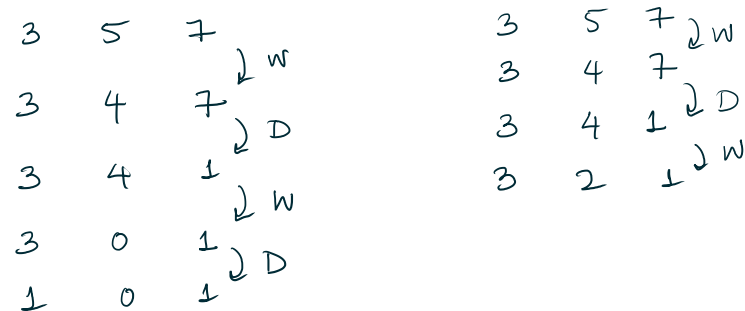
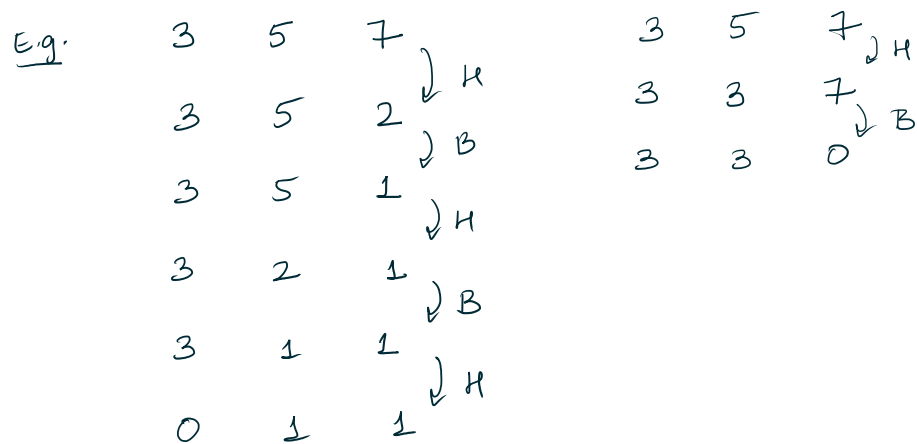
Claim: All positions of the form (n,n) are P positions:

Pf: (Base case): $(0,0)$ is a P position

Consider (n,n) : Every possible move sends you to a position of the form (n,m) for some $m < n$, then the next player can bring it back to (m,m) .
By induction, (m,m) is a P-position, so (n,m) is an N-position for each $m < n$.

\Rightarrow Every path from (n,n) moves to an N-position, so (n,n) is a P-position.

Harder case: 3 or more piles.



* The winning strategy (for any number of piles).

Suppose we have k piles of x_1, x_2, \dots, x_k berries each

We want to compute "the nim sum" of x_1, \dots, x_k to determine whether the position is P or N.

① Convert each number x_i into binary.

② XOR these numbers together.
(Add without carrying)

③ Look at the final number

* If the final answer equals 0, then it's a P position

* If the final answer is non-zero, it is an N position.

$$\begin{array}{r}
 3 \quad 5 \quad 7 \\
 \text{"} \quad \text{"} \quad \text{"} \\
 2+1 \quad 4+1 \quad 4+2+1 \\
 11_2 \quad 101_2 \quad 111_2 \\
 \\
 \oplus \quad 11_2 \\
 \oplus \quad 101_2 \\
 \oplus \quad 111_2 \\
 \hline
 001_2
 \end{array}$$

This is an N position
(Win for player 1)

E.g. $(5, 7)$

$$\begin{array}{c}
 \downarrow \quad \downarrow \\
 101_2 \quad 111_2
 \end{array}$$

$$\begin{array}{r}
 101 \\
 \oplus 111 \\
 \hline
 010
 \end{array}$$

N position

$(5, 5)$

$$\begin{array}{c}
 \downarrow \quad \downarrow \\
 101_2 \quad 101_2
 \end{array}$$

$$\begin{array}{r}
 101 \\
 \oplus 101 \\
 \hline
 000
 \end{array}$$

P position

E.g. 15 20 5 4 11 (5 piles)

* $15 = 8+4+2+1 = 1111_2$

* $20 = 16+4 = 10100_2$

* $5 = 101_2$

* $4 = 100_2$

* $11 = 8+2+1 = 1011_2$

$$\begin{array}{r}
 1111_2 \\
 10100_2 \\
 101_2 \\
 100_2 \\
 \oplus \quad \hline
 1011_2 \\
 10001_2
 \end{array}$$

$\neq 0$

[N position]