MATH 2301 22 Oct 2020

* **Strategic labelling:**
  - Draw a graph of all game positions, connecting $A \to B$ if there is a move from $A \to B$.
  - Label a position as "P" if the previous player wins
  - Label a position as "N" if the next player wins

  (see yesterday's notes)

* **The game of nim.**

  Some number of piles of berries.
  - You can eat any +ve number of berries from any one pile on your turn.
  - The person who eats the last berry wins.

* **Analysis.**

  Easiest case: There is only one pile of berries.

  The first player eats everything and wins!

  Medium difficulty case: Two piles.

  E.g. • $(2,3) \to$ N position (yesterday)
  • $(2,2) \to$ P position (yesterday)

  E.g.: $(5,7)$ berries

  You can use a "mirroring" strategy

  $\to$ Eat two berries out of 7 (player 1) $\to$ $(5,5)$

  Claim: All positions of the form (n,n) are P positions:

Pf: (Base case): (0,0) is a P position

Consider (n,n): Every possible move sends you to a position of the form (n,m) for some $m<n$, then the next player can bring it back to (m,m). By induction, (m,m) is a P-position, so (n,m) is an N-position for each $m<n$.

$\Rightarrow$ Every path from (n,n) moves to an N-position, so (n,n) is a P-position.

Harder case: 3 or more piles.

E.g.

3	5	7	
			↓ H
3	5	2	
			↓ B
3	5	1	
			↓ H
3	2	1	
			↓ B
3	1	1	
			↓ H
0	1	1	

3	5	7	↓ H
3	3	7	↓ B
3	3	0	

3	5	7	
			↓ W
3	4	7	
			↓ D
3	4	1	
			↓ W
3	0	1	
			↓ D
1	0	1	

3	5	7	↓ W
3	4	7	
3	4	1	↓ D
3	2	1	↓ W

* The winning strategy (for any number of piles).

Suppose we have k piles of $x_1, x_2, \cdots x_k$ berries each.

We want to compute "the nim sum" of $x_1, \cdots, x_k$ to determine whether the position is P or N.

1. Convert each number x_i into binary.
2. XOR these numbers together. (Add without carrying)
3. Look at the final number
   * If the final answer equals 0, then it's a P position
   * If the final answer is non-zero, it is an N position.

$3 = 2+1 = 11_2$, $5 = 4+1 = 101_2$, $7 = 4+2+1 = 111_2$

$$\begin{array}{rr} & 11_2 \\ \oplus & 101_2 \\ \oplus & 111_2 \\ \hline & 001_2 \end{array}$$

This is an N position (Win for player 1)

E.g. 15 20 5 4 11 (5 piles)

* $15 = 8+4+2+1 = 1111_2$
* $20 = 16+4 = 10100_2$
* $5 = 101_2$
* $4 = 100_2$
* $11 = 8+2+1 = 1011_2$

$$\begin{array}{rr} & 1111_2 \\ & 10100_2 \\ & 101_2 \\ & 100_2 \\ \oplus & 1011_2 \\ \hline & 10001_2 \end{array}$$

$\neq 0$

[N position]

E.g. $(5,7)$ → 101_2, 111_2

$$\begin{array}{r} 101 \\ 111 \\ \hline 010 \end{array}$$

N position

$(5,5)$ → 101_2, 101_2

$$\begin{array}{r} 101 \\ 101 \\ \hline 000 \end{array}$$

P positions