MATH 2301 23 Oct 2020

* Nim-sum.

Let x be a non-negative integer.

Then x has a unique binary representation.

We can find it by:

① Writing x as a sum of distinct powers of 2:

$$x = \sum_{j=0}^{n} a_j 2^j = a_n 2^n + a_{n-1} 2^{n-1} + \cdots + a_1 2^1 + a_0$$

$$x = (a_n a_{n-1} \cdots a_1 a_0)_2 \text{ or "binary"}$$

[Note: $a_i = 0$ or $a_i = 1$ for each i]

② In practice, we do this by successively subtracting the largest possible power of 2 from x until we reach 0.

E.g. $x = 25 = 16 + 9 = 16 + 8 + 1 = 2^4 + 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1$

$x = (11001)_2$

Let $x_1, x_2, \cdots x_n$ be non-negative integers.

Def: The nim-sum of $x_1, \ldots x_n$ is defined to be the XOR of the binary representations of $x_1, \cdots, x_n$.

It is denoted by $x_1 \oplus x_2 \oplus \cdots \oplus x_n$.

E.g. Consider 5, 8, 10.

$$\begin{array}{rcll} 5 &=& 4+1 &= \quad 101_2 \\ \oplus\ 8 &=& 8 &= \oplus\ 1000_2 \\ \oplus\ 10 &=& 8+2 &= \oplus\ 1010_2 \\ \hline & & & \quad\ 0111_2 \end{array}$$

Addition without carries, or

in each column, we add modulo 2.

$5 \oplus 8 \oplus 10 = 111_2 = 2^2 \cdot 1 + 2^1 \cdot 1 + 2^0 \cdot 1$

$= 4 + 2 + 1 = 7$

$5 \oplus 8 \oplus 10 = 7$

* Properties of nim-sum:

① It is associative: $(x_1 \oplus x_2) \oplus x_3 = x_1 \oplus (x_2 \oplus x_3)$

② It is commutative: $x_1 \oplus x_2 = x_2 \oplus x_1$

③ 0 is the identity: $0 \oplus x = x \oplus 0 = x$

④ Each number is its own inverse:

$x \oplus x = 0$ for any x!

E.g. $x = 11 = 8 + 2 + 1 = 1011$

$$\begin{array}{r} 1011_2 \\ \oplus\ 1011_2 \\ \hline 0000_2 \end{array}$$

* We see that the cancellation law holds for $\oplus$:

If $x \oplus y = x \oplus z$ then:

$x \oplus x \oplus y = x \oplus x \oplus z$ (omit parentheses by associativity)

$0 \oplus y = 0 \oplus z$

$\Rightarrow y = z$!

Thm: A position $(x_1, \ldots, x_n)$ in a game of nim is a P-position if and only if $x_1 \oplus \cdots \oplus x_n = 0$

It is an N-position if and only if $x_1 \oplus \cdots \oplus x_n \neq 0$.

[In other words, each position can be labelled by a "nimber", ie the nim-sum of the heap sizes. We can deduce P/N based on the nimber.]

E.g. $(13, 12, 8)$ ← labelled by the nimber 9.

$$\begin{aligned} 13 &= 8+4+1 &&= 1101_2 \\ \oplus\ 12 &= 8+4 &&= \oplus\ 1100_2 \\ \oplus\ 8 &= 8 &&= \oplus\ 1000_2 \\ \hline & && \ 1001_2 = 9 \end{aligned}$$

$9 \neq 0$, so this is an N-position (according to thm)

The theorem implies that we can move to a position whose nimber label (nim sum of heap sizes) equals 0.

Idea (Erin): Take 9 from 13 → left with $4 = 100_2$

$$\begin{array}{r} 100_2 \\ 1100_2 \\ 1000_2 \\ \hline 0000_2 \end{array}$$

that works! $(4, 12, 8)$ is a P-position.

Idea (Hanwen): 7 from 12:

$$\begin{array}{r} 1101_2 \\ 101_2 \\ 1000_2 \\ \hline 0000_2 \end{array}$$

works! $(13, 5, 8)$ is a p-position

Idea (Hamzah): 7 from 8:

$$\begin{array}{r} 1101_2 \\ 1100_2 \\ 1_2 \\ \hline 0000_2 \end{array}$$

works! $(13, 12, 1)$ is a P-position.

* We're trying to get an even number of 1s in each column.

Pf of theorem:

Let $(x_1, \dots, x_n)$ be a game position.

1. If $x_1 = x_2 = \cdots = x_n = 0$ then it's obviously a P position

2. Suppose not all piles are empty, and suppose $x_1 \oplus \cdots \oplus x_n \neq 0$.

Let's show that we can reach a position with nim-sum 0

E.g.

$$\begin{array}{r} 1101_2 \\ \oplus\ \boxed{\overset{0}{\cancel{1}}\,1\,0\,\overset{1}{\cancel{0}}_2} \\ \oplus\ 1000_2 \\ \hline 1001_2 \end{array}$$

1. Find the leftmost column with an odd number of 1s.
2. Choose one of the numbers that has a 1 in that column. This is the heap we'll modify.
3. Change this binary number by flipping some digits, so that each column ends up with an even number of 1s.

[in example, we changed 1100_2 to 0101_2]

Since we flipped the 1 to a 0 in the leftmost column with an odd number of 1s, this process ensures that the number decreases.

[in example we could have also done
$1101 \to 0100$ ie take 9 from 13, or
$1000 \to 0001$ ie take 7 from 8.]

In other words, take the chosen number & XOR with the original nim-sum

e.g. $1100 \to 1100 \oplus 1001 = 0101$.

To finish, we need to show that If $x_1 \oplus \cdots \oplus x_n = 0$ then every move sends you to a position with non-zero nim sum.

Suppose from $(x_1, \ldots, x_n) \to (x_1, \ldots, x_i', \ldots, x_n)$ with

$$x_i' < x_i$$

If $x_1 \oplus x_2 \oplus \cdots \oplus x_i' \oplus \cdots \oplus x_n = 0 = x_1 \oplus x_2 \oplus \cdots \oplus x_n$ then cancellation law implies that

$$x_i' = x_i \quad !$$

So this can't happen.

$\square$