

* Nim-sum.

Let x be a non-negative integer.

Then x has a unique binary representation.

We can find it by:

① Writing x as a sum of distinct powers of 2:

$$x = \sum_{j=0}^n a_j 2^j = a_n 2^n + a_{n-1} 2^{n-1} + \dots + a_1 2^1 + a_0$$

$$x = (a_n a_{n-1} \dots a_1 a_0)_2 \text{ or "binary"}$$

[Note:
 $a_i = 0$ or
 $a_i = 1$
 for each i]

② In practice, we do this by successively subtracting the largest possible power of 2 from x until we reach 0.

E.g. $x = 25 = 16 + 9 = 16 + 8 + 1 = 2^4 + 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1$

$$x = (11001)_2$$

Let x_0, x_1, \dots, x_n be non-negative integers.

Def: The nim-sum of x_1, \dots, x_n is defined to be the XOR of the binary representations of x_1, \dots, x_n .

It is denoted by $x_1 \oplus x_2 \oplus \dots \oplus x_n$.

E.g. Consider 5, 8, 10.

$$5 = 4 + 1 = 101_2$$

$$\oplus 8 = 8 = \oplus 1000_2$$

$$\oplus 10 = 8 + 2 = \oplus \begin{array}{r} 1010_2 \\ \hline 0111_2 \end{array}$$

Addition without carries, or

in each column, we add modulo 2.

$$5 \oplus 8 \oplus 10 = 111_2 = 2^2 \cdot 1 + 2^1 \cdot 1 + 2^0 \cdot 1$$

$$= 4 + 2 + 1 = 7$$

$$5 \oplus 8 \oplus 10 = 7$$

* Properties of nim-sum.

① It is associative: $(x_1 \oplus x_2) \oplus x_3 = x_1 \oplus (x_2 \oplus x_3)$

② It is commutative: $x_1 \oplus x_2 = x_2 \oplus x_1$

③ 0 is the identity: $0 \oplus x = x \oplus 0 = x$

④ Each number is its own inverse:

$$x \oplus x = 0 \text{ for any } x!$$

E.g. $x = 11 = 8 + 2 + 1 = 1011$

$$\begin{array}{r} 1011_2 \\ \oplus 1011_2 \\ \hline 0000_2 \end{array}$$

* We see that the cancellation law holds for \oplus :

If $x \oplus y = x \oplus z$ then:

$$x \oplus x \oplus y = x \oplus x \oplus z$$

(omit parentheses by associativity)

$$0 \oplus y = 0 \oplus z$$

$$\Rightarrow y = z!$$

Thm: A position (x_1, \dots, x_n) in a game of nim

is a P-position if and only if $x_1 \oplus \dots \oplus x_n = 0$

It is an N-position if and only if $x_1 \oplus \dots \oplus x_n \neq 0$.

[In other words, each position can be labelled by a

"number", i.e. the nim-sum of the heap sizes.

We can deduce P/N based on the number.]

E.g. $(13, 12, 8)$ ← labelled by the number 9.

$$13 = 8 + 4 + 1 = 1101_2$$

$$\oplus 12 = 8 + 4 = \oplus 1100_2$$

$$\oplus 8 = 8 = \oplus \underline{1000_2}$$

$$1001_2 = \textcircled{9}$$

$9 \neq 0$, so this is an N-position (according to Hui)

The theorem implies that we can move to a position whose number label (nim sum of heap sizes) equals 0.
(ENN)

Idea: Take 9 from 13 → left with $4 = 100_2$

$$\begin{array}{r} 100_2 \\ 1100_2 \\ \underline{1000_2} \\ 0000_2 \end{array}$$

that works!

$(4, 12, 8)$ is a P-position.

Idea (Hamwen): 7 from 12:

$$\begin{array}{r} 1101_2 \\ 101_2 \\ \underline{1000_2} \\ 0000_2 \end{array}$$

works!

$(13, 5, 8)$ is a P-position

Idea (Hamzah): 7 from 8:

$$\begin{array}{r} 1101_2 \\ 1100_2 \\ \underline{1_2} \\ 0000_2 \end{array}$$

works!

$(13, 12, 1)$ is a P-position.

* We're trying to get an even number of 1s in each column.

Pf of theorem:

Let (x_1, \dots, x_n) be a game position.

① If $x_1 = x_2 = \dots = x_n = 0$ then it's obviously a P position

② Suppose not all piles are empty, and suppose $x_1 \oplus \dots \oplus x_n \neq 0$.

Let's show that we can reach a position with nim-sum 0.

E.g.

$$\begin{array}{r} \downarrow \\ 1101_2 \\ \oplus \boxed{1100_2} \\ \oplus \underline{1000_2} \\ 1001_2 \end{array}$$

① Find the leftmost column with an odd number of 1s.

② Choose one of the numbers that has a 1 in that column.

This is the heap we'll modify.

③ Change this binary number by flipping some digits, so that each column ends up with an even number of 1s.

[In example, we changed 1100_2 to 0101_2]

Since we flipped the 1 to a 0 in the leftmost column with an odd number of 1s, this process ensures that the number decreases.

[In example we could have also done

$$\begin{array}{l} 1101 \rightarrow 0100 \text{ ie take 9 from 13, or} \\ 1000 \rightarrow 0001 \text{ ie take 7 from 8.} \end{array}$$

In other words, take the chosen number & XOR with the original nim-sum

$$\text{e.g. } 1100 \rightarrow 1100 \oplus 1001 = 0101.$$

To finish, we need to show that if $x_1 \oplus \dots \oplus x_n = 0$
then every move sends you to a position with non-zero
nim sum.

Suppose from $(x_1, \dots, x_n) \rightarrow (x_1, \dots, x'_i, \dots, x_n)$ with

$$x'_i < x_i$$

$$\text{If } x_1 \oplus x_2 \oplus \dots \oplus x'_i \oplus \dots \oplus x_n = 0 = x_1 \oplus x_2 \oplus \dots \oplus x_i \oplus \dots \oplus x_n$$

then cancellation law implies that

$$x'_i = x_i \quad ! \quad \text{So this can't happen.}$$

□