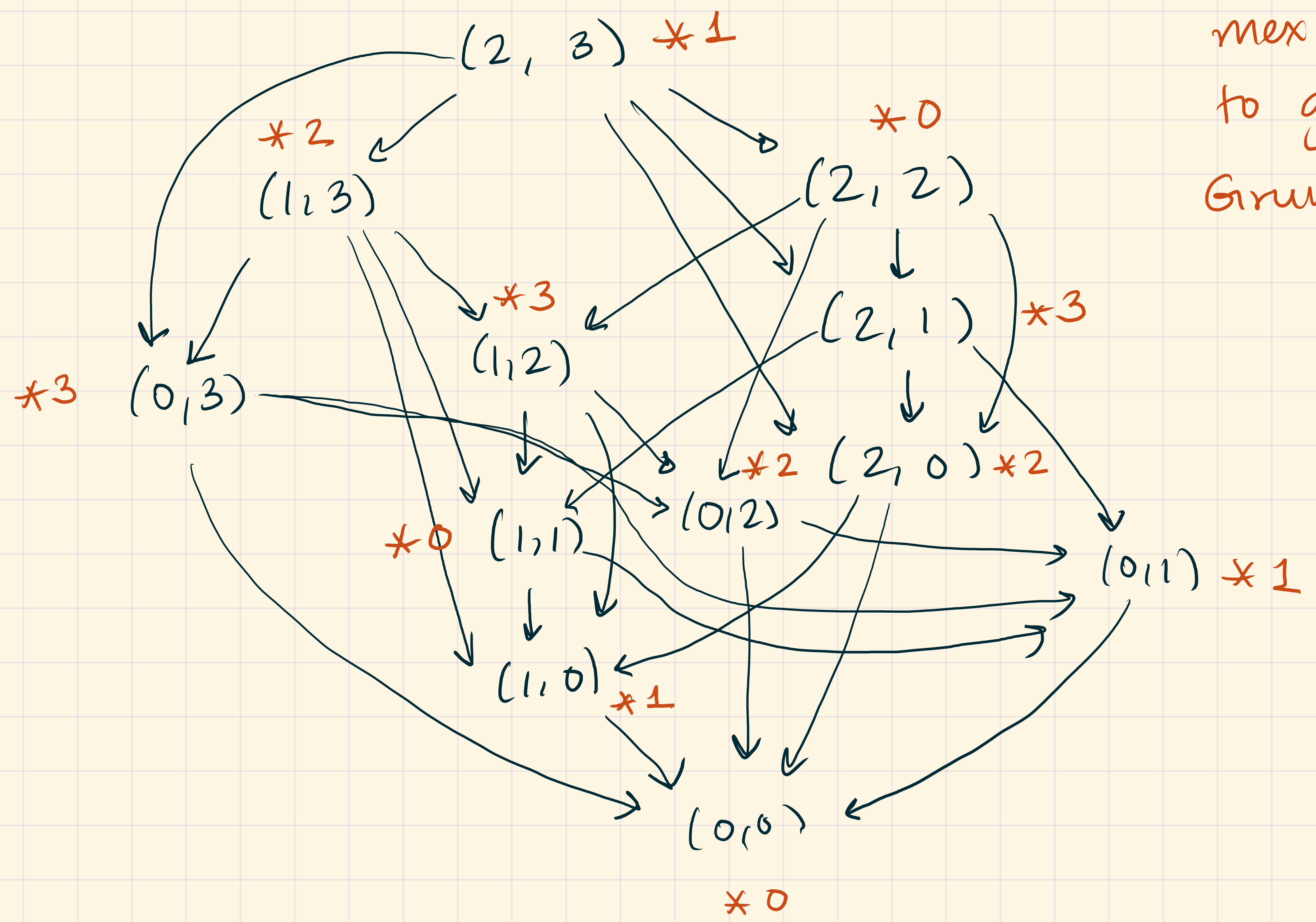
MATH 2301 29 Oct 2020

* Yesterday: We discussed Grundy labelling & the mex rule.

Example: (2,3) nim.

Label using mex rule to get Grundy labels.



Observation: The Grundy label for a nim position is exactly the same as its nimber label!

E.g. $1 \oplus 2 = \begin{array}{r} 1_2 \\ \oplus\, 10_2 \\ \hline 11_2 \end{array} = 3$ } Nimber label is $*3$

= Grundy label!

In fact, this is a theorem!

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* Operations on impartial combinatorial games. (à la Conway)

- Addition of games: We'll use letters like G & H to denote games.

Formally, a game G is either:

(1) $G = \phi$ (i.e. a terminal position, where you can't make any other moves)

(2) $G = \{G_1, \dots, G_k\}$ where $G_1, \dots, G_k$ are also games

(i.e. from G, the possible moves are $G_1, G_2, \dots, G_k$)

E.g. $(1, 1)$ nim

$(1,1) \longrightarrow \{\text{game represented by } (1,0)\} = \{\{\phi\}\}$

$$
\begin{array}{ccc}
 & (1,1) & \\
\swarrow & & \searrow \\
(0,1) & = & (1,0) \\
\searrow & & \swarrow \\
 & (0,0) &
\end{array}
$$

$(1,0) \longleftrightarrow \{\text{game representing } (0,0)\} = \{\phi\}$

$(0,0) \longleftrightarrow \phi$ game

Addition of games:

Let G_1 and G_2 be games.

We define $G_1 + G_2$ as the following game:

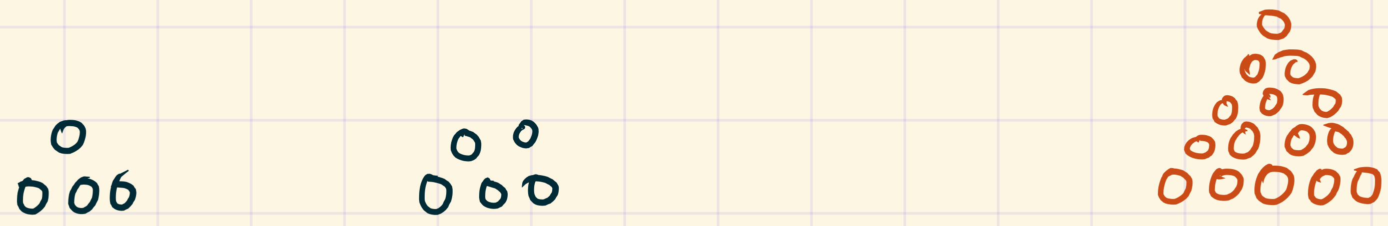
informally, you play the two games "simultaneously":

a legal move is a legal move either in G_1 or in G_2. (one at a time).

E.g. The $(3,4)$ game of nim is the sum:

(nim with a pile size of 3) + (nim with pile size of 4)

E.g. (nim with pile sizes of 4, 5) + (subtraction game for $S = \{2, 3\}$ & pile size 10)



A legal move is either:

(1) Take any number of berries from either pile 1 or 2,

or

(2) Take either 2 or 3 berries from pile 3.

* Some properties of game addition:

(1) It is commutative

(2) It is associative

* The game ϕ is distinguished: any terminal position in any game is represented by ϕ.

(3) If G is any game, then $G + \phi = \phi + G = G$.

i.e, ϕ is the additive identity.

(You can't make any moves in ϕ, so $G+\phi$ means you're just playing in G.)

* Each game G has a Grundy label, and it also has a P/N label. or "the outcome class"

Eg. ϕ has a Grundy label of $*0$

and a P/N label of P

Q: Given the outcome classes and Grundy labels of G & H, can we figure out the outcome class & Grundy label of $G+H$?

Examples

* Given any G, we can answer this question for $H = \emptyset$

  The outcome class of $G + \emptyset$ is the same as for G

  Grundy label of $G + \emptyset$ is the same as for G

* Let G be any game.

  Consider $G + G$. : the outcome class must be P:
  the winning strategy for the second player is just to copy the first player's move, in the other copy of the game.

  [Similar to the strategy for (n, n) nim.]