MATH 2301 30 Oct 2020

* Recap: We have seen:

(1) The winning strategy for nim

(2) Nimber labelling for nim-positions (using XOR)

(3) Grundy labelling for any game position $\Rightarrow$ outcome class

(4) Nimber = Grundy label for a nim position

(5) Formal addition of games:

$G+H$ means we play G & H in parallel, making a move in any one game at a time.

Q: If we know the outcome classes / Grundy labels of G & H, what can we say about $G+H$?

Eg. • If $H=\emptyset$ (no possible moves) then $G+\emptyset$ has the same Grundy label & outcome class as G.

• If $H=G$ then $G+G$ has an outcome class of P [and therefore, a Grundy label of $*0$]

* Knowing just the outcome classes of G & H, we can't usually say anything about the outcome class of $G+H$.

E.g. (1) $G =$ nim game (3), $H =$ nim game (2) (both N) } $G+H$ also N, has nimber/Grundy label $*1$

(2) $G=H=$ nim game (3) (both N) } $G+H = G+G$ is P.

* An equivalence relation on games.

Defn: Say that $G \approx G'$ if: for any game H, the games $G+H$ and $G'+H$ have the same outcome class.

Check:

(1) $G\approx G$ for each G? Clearly true.

(2) $G\approx G'$ then $G'\approx G$? Also clearly true.

(3) If $G\approx G'$ & $G'\approx G''$ then $G\approx G''$?

Let H be any game.

$G+H$ has the same outcome class as $G'+H$

$G'+H$ has the same outcome class as $G''+H$

$\Rightarrow$ $G+H$ has the same outcome class as $G''+H$

(proves transitivity)

Some properties:

(1) Suppose $G\approx G'$ then they must have the same outcome class. (take $H=\emptyset$ in the def.)

(2) Suppose $G\approx G'$. then $G+G'$ is a P game.

If $H=G'$, we have $G+G'$ & $G'+G'$ have the same outcome class & $G'+G'$ is a P game

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Examples

(1) $G=$ nim game (2), $G'=$ nim game (3) } take $H=G$

$G+H$ is a P game (nimber $10_2 \oplus 10_2 = *0$)

$G'+H$ is an N game (nimber $11_2 \oplus 10_2 = 01_2 = *1$)

(2) $G =$ nim game (2)
$G' =$ nim game (2, 2, 2)
} Is $G \approx G'$??

- Is it true that $G+H$ & $G'+H$ have the same outcome class? $\to$ seems difficult to check

- Is it true if H is any other nim game?

  In this case we can compute the nimber for H, suppose that is $*h$.

  $G+H$ has nimber $*2 \oplus *h$

  $G'+H$ has nimber $\underbrace{(*2 \oplus *2)}_{=*0} \oplus *2 \oplus *h$

$$= *2 \oplus *h.$$

* Fact: In fact, $G \approx G'$.

Theorem: $G \approx G'$ if and only if $G+G'$ is a P-game.

E.g. $G =$ nim (2)
$G' =$ nim (2, 2, 2)
} $G+G'$ has nim-sum $*0$ is a P-game

By the theorem, $G \approx G'$.

E.g. $G =$ nim (3, 1) $\to 11_2 \oplus 01_2 = 10_2 = *2$

$G' =$ nim (7, 5) $\to 111_2 \oplus 101_2 = 10_2 = *2$.

$\Rightarrow G \approx G'$ by theorem & $G \approx$ nim (2)

E.g. $G =$ nim (5, 10, 7) = nim sum $*8$

$$\begin{array}{rr} \to & 101_2 \\ & 1010_2 \\ \oplus & 111_2 \\ \hline & 1000_2 \end{array}$$

$G' =$ nim (8) $\Rightarrow G \approx G'$

Corollary: Any nim game G is equivalent to a nim game with a single pile of berries; namely with size = nimber of G.

Let G be any (impartial, combinatorial) game.

Recall that G has a Grundy label.

Thm: Let G & H be two games with Grundy labels $*g$ & $*h$.

Then $G+H$ has Grundy label $*g \oplus *h$
nim-sum!

Example/Exercise: $G =$

*		

chomp

$H =$ subtraction game with $S = \{1, 2\}$ & $n = 5$

The Grundy label of the combined game $G+H$ is going to be $*g + *h$

<u>Thm</u> (Sprague-Grundy theorem): Let G be any game. Let $*g$ be its Grundy label. Then $G \approx G'$, where G' is a nim game with a single pile of size g!

$\Rightarrow$ EVERY impartial combinatorial game is equivalent to nim!