MATH 2301 30 Oct 2020

* Recap: We have seen:

(1) The winning strategy for nim

(2) Nimber labelling for nim-positions (using XOR)

(3) Grundy labelling for any game position ⇒ outcome class

(4) Nimber = Grundy label for a nim position

(5) Formal addition of games:

G+H means we play G & H in parallel, making a move in any one game at a time.

Q: If we know the outcome classes / Grundy labels of G & H, what can we say about G+H?

Eg. • If $H = \phi$ (no possible moves) then $G+\phi$ has the same Grundy label & outcome class as G.

• If H = G then G+G has an outcome class of P [and therefore, a Grundy label of $*0$]

* Knowing just the outcome classes of G & H, we can't usually say anything about the outcome class of G+H.

E.g. (1) G = nim game (3)
H = nim game (2)
(both N)

G+H also N, has nimber/Grundy label $*1$

(2) G = H = nim game (3)
(both N)

G+H = G+G is P.

* An equivalence relation on games.

<u>Defn</u>: Say that $G \approx G'$ if:
for any game H, the games $G+H$ and $G'+H$ have the same outcome class.

<u>Check</u>:

(1) $G \approx G$ for each G? Clearly true.

(2) $G \approx G'$ then $G' \approx G$? Also clearly true.

(3) If $G \approx G'$ & $G' \approx G''$ then $G \approx G''$?

Let H be any game.

$G+H$ has the same outcome class as $G'+H$

$G'+H$ has the same outcome class as $G''+H$

$\Rightarrow$ $G+H$ has the same outcome class as $G''+H$

(proves transitivity)

Some properties:

(1) Suppose $G \approx G'$ then they must have the same outcome class. (take $H = \phi$ in the def.)

(2) Suppose $G \approx G'$. then $G + G'$ is a P game.

If $H = G'$, we have $G+G'$ & $G'+G'$ have the same outcome class & $G'+G'$ is a P game

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<u>Examples</u>

(1) $G =$ nim game (2)
$G' =$ nim game (3) } take $H = G$

$G+H$ is a P game (nimber $10_2 \oplus 10_2 = *0$)

$G'+H$ is an N game (nimber $11_2 \oplus 10_2 = 01_2 = *1$)

(2) $\left.\begin{array}{l} G = \text{nim game}(2) \\ G' = \text{nim game}(2,2,2) \end{array}\right\}$ Is $G \approx G'$??

- Is it true that $G+H$ & $G'+H$ have the same outcome class? → seems difficult to check

- Is it true if H is any other nim game?

  In this case we can compute the nimber for H, suppose that is $*h$.

  $G+H$ has nimber $*2 \oplus *h$

  $G'+H$ has nimber $\underbrace{(*2 \oplus *2)}_{= *0} \oplus *2 \oplus *h$

  $= *2 \oplus *h$.

* Fact: In fact, $G \approx G'$.

Theorem: $G \approx G'$ if and only if $G+G'$ is a P-game.

E.g. $\left.\begin{array}{l} G = \text{nim}(2) \\ G' = \text{nim}(2,2,2) \end{array}\right\}$ $G+G'$ has nim-sum $*0$ is a P-game

By the theorem, $G \approx G'$.

E.g. $G = \text{nim}(3,1) \rightarrow 11_2 \oplus 01_2 = 10_2 = *2$

$G' = \text{nim}(7,5) \rightarrow 111_2 \oplus 101_2 = 10_2 = *2$.

$\Rightarrow G \approx G'$ by theorem & $G \approx \text{nim}(2)$

E.g. $G = \text{nim}(5, 10, 7) = \text{nim sum } *8$

$$
\begin{array}{rr}
\rightarrow & 101_2 \\
 & 1010_2 \\
\oplus & 111_2 \\
\hline
 & 1000_2
\end{array}
$$

$G' = \text{nim}(8) \Rightarrow G \approx G'$

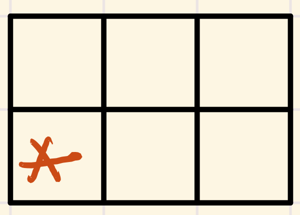
Corollary: Any nim game G is equivalent to a nim game with a single pile of berries; namely with size = nimber of G.

Let G be any (impartial, combinatorial) game. Recall that G has a Grundy label.

Thm: Let G & H be two games with Grundy labels $*g$ & $*h$.

Then $G + H$ has Grundy label $*g \oplus *h$

nim-sum!

Example/: Exercise $G =$  chomp

$H =$ subtraction game with $S = \{1, 2\}$ & $n = 5$

The Grundy label of the combined game $G+H$ is going to be $*g + *h$

Thm (Sprague-Grundy theorem): Let G be any game. Let $*g$ be its Grundy label.
Then $G \approx G'$, where G' is a nim game with a single pile of size g!

$\Rightarrow$ EVERY impartial combinatorial game is equivalent to nim!