Math 2301

* From last time: functions

A function is a relation $R \subseteq S \times T$ satisfies two additional properties:

(1) If $(s, t_1) \in R$ and $(s, t_2) \in R$ for some $s \in S$ and $t_1, t_2 \in T$, then $t_1 = t_2$
(unique "output" for each "input" that appears)

(2) [Conventionally] for every $s \in S$, there is a $t \in T$ such that $(s, t) \in R$.
(for any "input" s, there is an "output" t)

In this case, we say that S is the domain & T is the codomain, and usually write functions as $f : S \to T$.

** Example

$\sin : \mathbb{R} \longrightarrow [-1, 1]$

name of our function

set of real numbers (domain)

(codomain)

$\text{square} = x^2 : \mathbb{N} \longrightarrow \mathbb{N}$

* Graphs

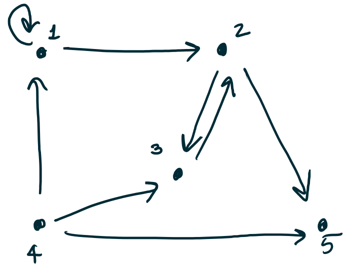
- Used to organise information visually
- Also an excellent computational tool.

** Definition: A directed graph consists of a set V of "vertices"/"nodes", and a binary relation on V, called $E \subseteq V \times V$. The elements of E are ordered pairs of vertices, called "edges".

** Drawing directed graphs:

$V = \{1, 2, 3, 4, 5\}$

$E = \{(1,1), (1,2), (2,3), (2,5), (3,2), (4,1), (4,3), (4,5)\}$



* Warning: Renumbering the vertices in a drawing, you'll get a graph that looks the same, but has a different V and a different E.

Technically, you'll get a different graph, but it's isomorphic to the first one.
More about this later.

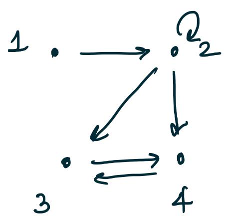
$\rightsquigarrow$ Many (V, E) pairs give you the same drawing

- At the moment, our definition says there'll be at most one edge in a given direction between two vertices.

** The adjacency matrix

Recall: A matrix is a rectangular array of numbers (typically).

Consider a graph $G = (V, E)$. Choose an ordering on the set V.



Let n be the number of elements in V.

Consider an $(n \times n)$ matrix
rows ↑ ↑ columns

end

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\text{start} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{array}
\longleftarrow \text{records edges from 1 to other vertices}
$$

Defn: Let (V, E) be a graph with n vertices (ordered). The adjacency matrix is an $n \times n$ matrix whose $(i,j)^{th}$ entry is the number of edges from i to j.

* Properties of relations (vis-a-vis the adjacency matrix).

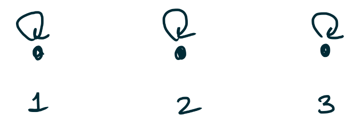
** Fix a set S, and the relation R will be on S.

** Reflexivity: A relation is called reflexive if for each $s \in S$, the pair $(s,s) \in R$

*** Example

$S = \{1, 2, 3\}$, $R = \{(1,1), (2,2), (3,3)\}$
"$x = y$"

*** Graph



Q [workshop]
What property should the adjacency matrix satisfy?

** Symmetry

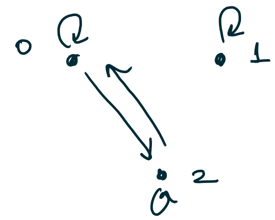
Defn: R is symmetric if whenever $(x,y) \in R$, we have $(y,x) \in R$.

*** Example

$S = \{0, 1, 2\}$; $(x, y) \in R$ if $x + y$ is even.

(Equivalently, $y + x$ is even, because $x + y = y + x$.)

*** Graph



Q: [workshop]

Adjacency matrix?

** Anti-symmetry [bad name !!]

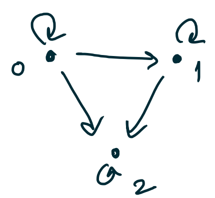
Defn: R is anti-symmetric if whenever $(x,y) \in R$ and $x \neq y$, the pair $(y,x) \notin R$.

Warning: A relation can be both symmetric and anti-symmetric! → workshop.

*** Example

$S = \{0, 1, 2\}$, $(x,y) \in R$ if $x \leq y$.

*** Graph



Q [workshop]

Adjacency matrix?

** Being a function

*** Example

$f: \{1,2,3\} \to \{1,2,3\}$

$f(x) = 4 - x$

*** Graph