Math 2301

* Properties of relations (continued)

Let R be a relation on a set S.

** Transitivity : We say that R is transitive if whenever $(x,y) \in R$ and $(y,z) \in R$, we also have $(x,z) \in R$.

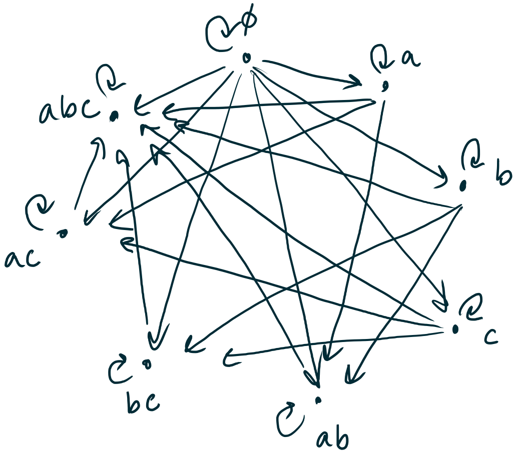
*** Example

S any set. We have a relation R on $P(S)$, where $(A,B) \in R$ if $A \subseteq B$.

If $A \subseteq B$ & $B \subseteq C$, then $A \subseteq C$.

*** Graph

$S = \{a, b, c\}$. $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$



Q : [workshop?]
What does the adjacency matrix look like

** Equivalence relations

Let R be a binary relation on a set S.

*** Definition : We say that R is an equivalence relation if it is reflexive, symmetric, and transitive.

* Equivalence relation generalise the idea of equality.

*** Examples and non-examples

- R on $\mathbb{Z}$ defined as

$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x+y \text{ is even}\}$

reflexive ✓
symmetric ✓
transitivity ✓

$x+y = 2k$ for some $k \in \mathbb{Z}$
$y+z = 2l$ for some $l \in \mathbb{Z}$
$x+z = 2k+2l-2y$ ← even

- R on $\mathbb{Z}$ defined as

$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x+y \text{ is odd}\}$

not reflexive !!
symmetric ✓
transitive !!

- R on $\mathbb{Z}$ defined as

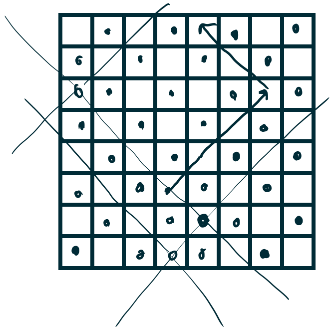
$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x-y \text{ is an integer multiple of } 17\}$

reflexive ✓
symmetric ✓
transitive ? ✓

$x-y = 17k$
$y-z = 17l$
$\Rightarrow x-z = 17(k+l)$ ✓

- R on $S := \{$ squares on a chessboard $\}$;

$R = \{(s_1, s_2) \mid s_2$ is reachable from s_1 via a sequence of bishop moves$\}$. [bishops can move any number of squares in a diagonal straight line]



reflexive ✓
symmetric ✓
transitive ✓

- $\{(s_1, s_2) \mid s_2$ is reachable from s_1 by at most a single bishop move$\}$

reflexive ✓
symmetric ✓
not transitive

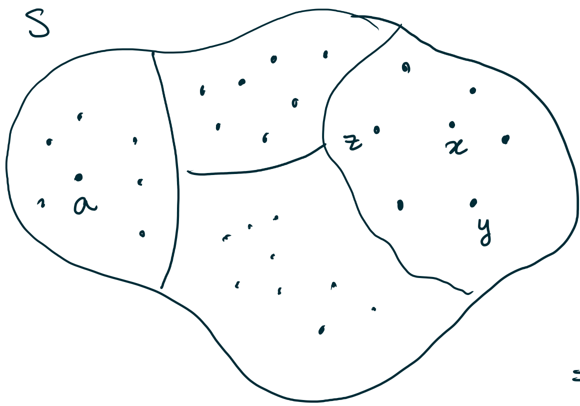
- $\{(s_1, s_2) \mid s_2$ is reachable from s_1 by at most two bishop moves$\}$

reflexive ✓
symmetric ✓
transitive? ✓

Suppose $(x, y) \in R$ & $(y, z) \in R$
$\Rightarrow x$ & z are the same colour $\Rightarrow z$ is reachable from x by ≤ 2 bishop moves.

** Equivalence classes

Notation: Let R be an equivalence relation on S. If $(x, y) \in R$, we usually write $x \sim_R y$, or simply $x \sim y$ if there is no confusion.
We'll often just shorten by saying "let $\sim$ be an equivalence relation".



Fix $x \in S$
Collect all $y \in S$ such that $x \sim y$.
If $x \sim y$ & $x \sim z$
then: $z \sim x$ (symmetry)
$\Rightarrow z \sim y$ (transitivity)

*** Definition: Let $\sim$ be an equivalence relation on S. Let $x \in S$. The equivalence class of x under $\sim$ is the set of all $y \in S$ such that $x \sim y$.

Usually denoted $[x]$ ← is a subset of S.

$$[x] = \{y \in S \mid x \sim y\}$$

*** <u>Proposition</u>

(1) Let $y \in [x]$. Then $x \in [y]$ and $[x] = [y]$

(2) If E_1 and E_2 are two equivalence classes, then either $E_1 = E_2$ or $E_1 \cap E_2 = \phi$.

<u>Proof</u>

(1) Let $y \in [x] \Rightarrow x \sim y$.

By symmetry, we have $y \sim x$

So, $x \in [y]$.

Let $y \in [x]$. $\Rightarrow x \sim y$

To show that $[x] = [y]$, suppose that $z \in [x]$

$\Rightarrow x \sim z$

By symmetry & transitivity, $y \sim z$

$\Rightarrow z \in [y]$

(Similarly if $z \in [y]$ then $z \in [x]$)

$\Rightarrow [x] = [y]$.

(2) Let E_1, E_2 be two classes

Suppose $E_1 \neq E_2$.

What if $E_1 \cap E_2 \neq \phi$?

finish next time.

If $z \in E_1 \cap E_2$, then $z \in E_1$ & $z \in E_2$.