

Math 2301

* Properties of relations (continued)

Let R be a relation on a set S .

** Transitivity: We say that R is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, we also have $(x, z) \in R$.

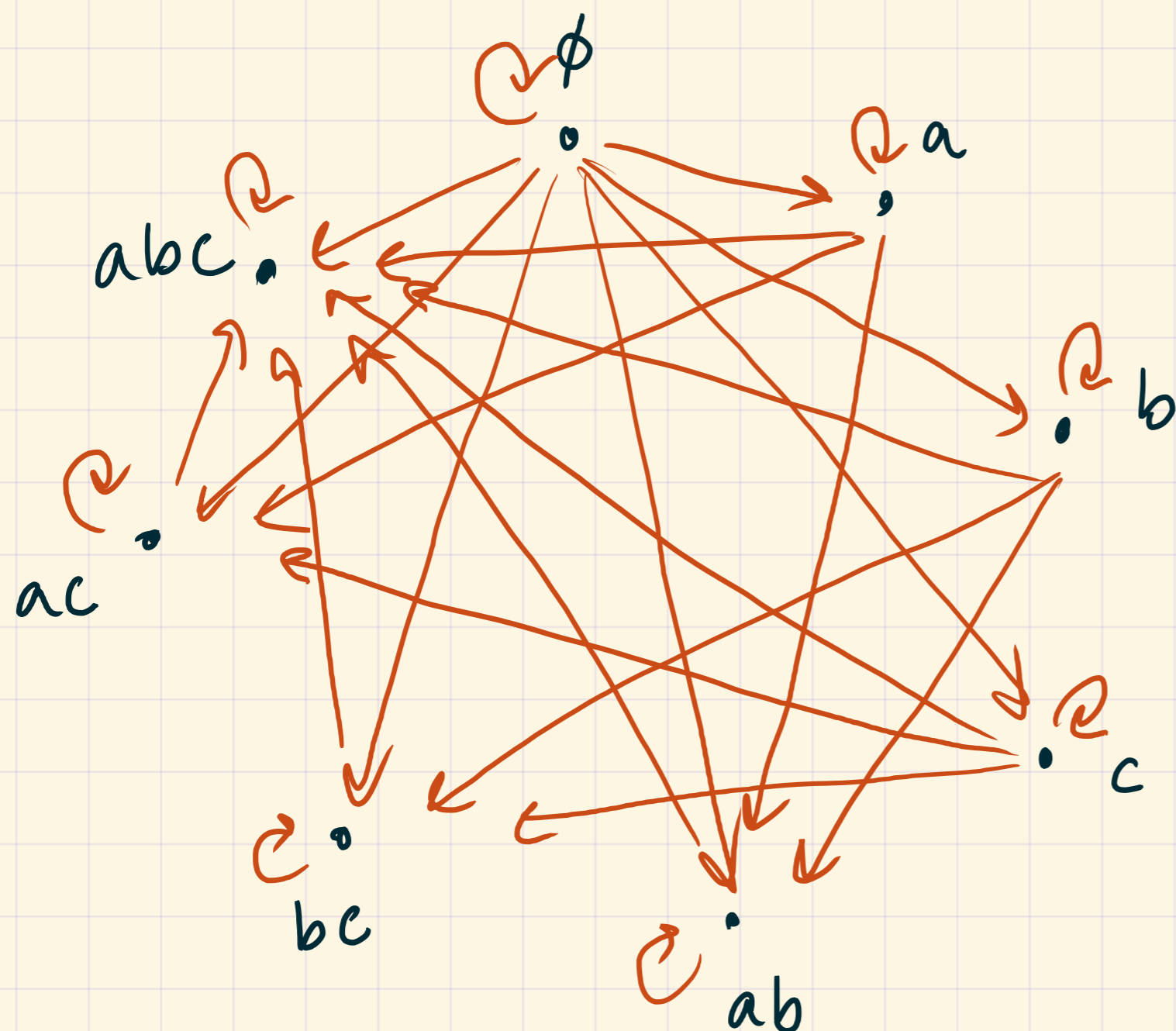
*** Example

S any set. We have a relation R on $\mathcal{P}(S)$, where $(A, B) \in R$ if $A \subseteq B$.

If $A \subseteq B$ & $B \subseteq C$, then $A \subseteq C$.

*** Graph

$S = \{a, b, c\}$. $\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$



Q: [workshop?]
What does the adjacency matrix look like

** Equivalence relations

Let R be a binary relation on a set S .

*** Definition: We say that R is an equivalence relation if it is reflexive, symmetric, and transitive.

* Equivalence relation generalise the idea of equality.

*** Examples and non-examples

- R on \mathbb{Z} defined as

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y \text{ is even}\}$$

reflexive ✓

symmetric ✓

transitivity ✓

$$\begin{aligned} x + y &= 2k \text{ for some } k \in \mathbb{Z} \\ y + z &= 2l \text{ for some } l \in \mathbb{Z} \\ \hline x + z &= 2k + 2l - 2y \leftarrow \text{even} \end{aligned}$$

- R on \mathbb{Z} defined as

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y \text{ is odd}\}$$

not reflexive ✗

symmetric ✓

transitive ✗

- R on \mathbb{Z} defined as

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is an integer multiple of } 17\}$$

reflexive ✓

symmetric ✓

transitive? ✓

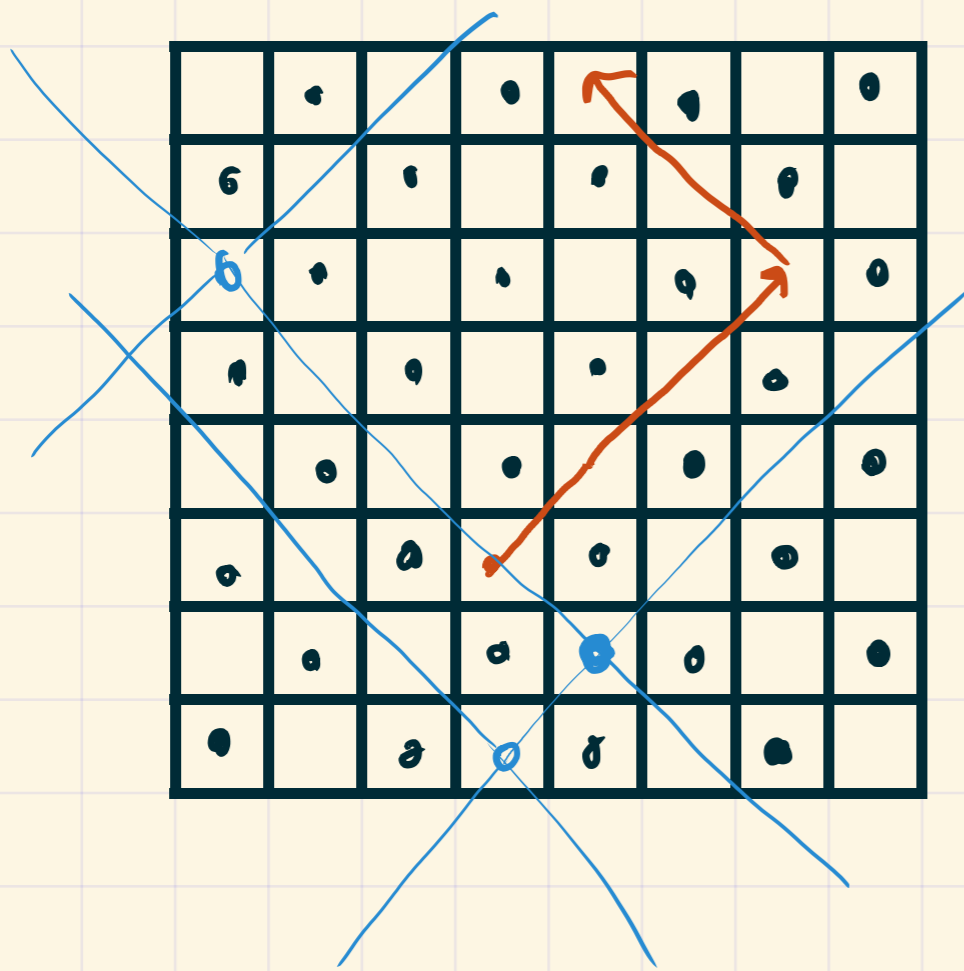
$$x - y = 17k$$

$$y - z = 17l$$

$$\Rightarrow x - z = 17(k+l) \checkmark$$

- R on $S := \{\text{squares on a chessboard}\}$;

$R = \{(s_1, s_2) \mid s_2 \text{ is reachable from } s_1 \text{ via a sequence of bishop moves}\}$. [bishops can move any number of squares in a diagonal straight line]



reflexive ✓

symmetric ✓

transitive ✓

- $\{(s_1, s_2) \mid s_2 \text{ is reachable from } s_1 \text{ by at most a single bishop move}\}$

reflexive ✓

symmetric ✓

not transitive

- $\{(s_1, s_2) \mid s_2 \text{ is reachable from } s_1 \text{ by at most two bishop moves}\}$

reflexive ✓

symmetric ✓

transitive? ✓

Suppose $(x, y) \in R$ &

$(y, z) \in R$

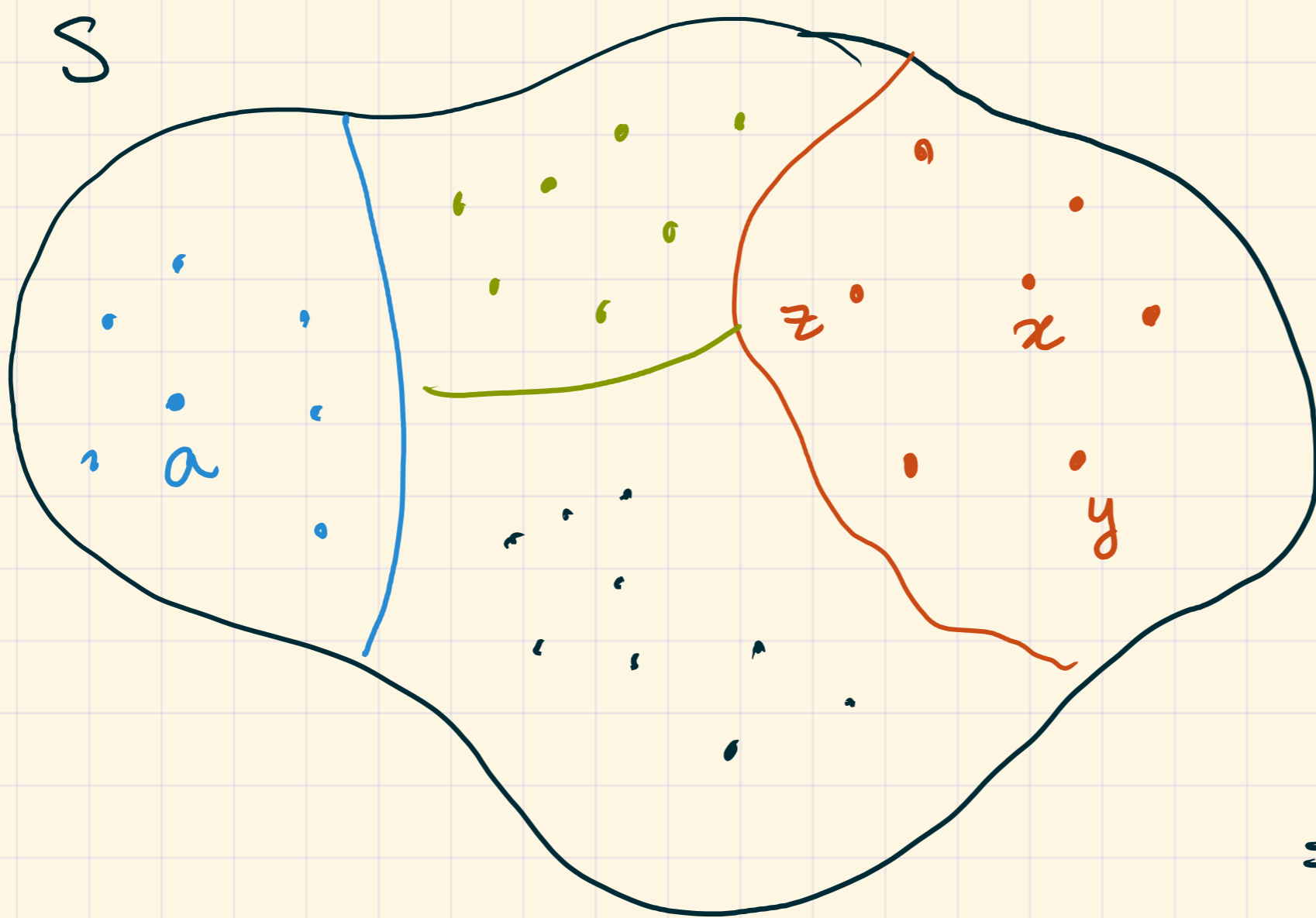
$\Rightarrow x$ & z are the same

colour $\Rightarrow z$ is reachable from x by ≤ 2 bishop moves.

** Equivalence classes

Notation: Let R be an equivalence relation on S . If $(x, y) \in R$, we usually write $x \sim_R y$, or simply $x \sim y$ if there is no confusion.

We'll often just shorten by saying "let \sim be an equivalence relation".



Fix $x \in S$

Collect all $y \in S$

such that

$x \sim y$.

If $x \sim y$ &

$x \sim z$

then: $z \sim x$ (symmetry)

$\Rightarrow z \sim y$ (transitivity)

*** Definition: Let \sim be an equivalence relation on S . Let $x \in S$. The equivalence class of x under \sim is the set of all $y \in S$ such that $x \sim y$.

Usually denoted $[x]$ \leftarrow is a subset of S .

$$[x] = \{y \in S \mid x \sim y\}$$

*** Proposition

(1) Let $y \in [x]$. Then $x \in [y]$ and $[x] = [y]$

(2) If E_1 and E_2 are two equivalence classes, then either $E_1 = E_2$ or $E_1 \cap E_2 = \emptyset$.

Proof

(1) Let $y \in [x] \Rightarrow x \sim y$.

By symmetry, we have $y \sim x$

So, $x \in [y]$.

Let $y \in [x] \Rightarrow x \sim y$

To show that $[x] = [y]$, suppose

that $z \in [x]$

$\Rightarrow x \sim z$

By symmetry & transitivity, $y \sim z$

$\Rightarrow z \in [y]$

(Similarly if $z \in [y]$ then $z \in [x]$)

$\Rightarrow [x] = [y]$.

(2) Let E_1, E_2 be two classes

Suppose $E_1 \neq E_2$.

What if $E_1 \cap E_2 \neq \emptyset$?

If $z \in E_1 \cap E_2$, then $z \in E_1$ & $z \in E_2$.

finish
next
time.