

MATH 2301

** Office hour: Monday 1-2 (in person + Zoom)

* Last time: Equivalence classes

*** Proposition

(1) Let $y \in [x]$. Then $x \in [y]$ and $[x] = [y]$

(2) If E_1 and E_2 are two equivalence classes, then either $E_1 = E_2$ or $E_1 \cap E_2 = \emptyset$.

Proof

(1) Let $y \in [x] \Rightarrow x \sim y$.

By symmetry, we have $y \sim x$

So, $x \in [y]$.

Let $y \in [x] \Rightarrow x \sim y$

To show that $[x] = [y]$, suppose

that $z \in [x]$

$\Rightarrow x \sim z$

By symmetry & transitivity, $y \sim z$

$\Rightarrow z \in [y]$

(Similarly if $z \in [y]$ then $z \in [x]$)

$\Rightarrow [x] = [y]$.

(2) If E_1 and E_2 are two equivalence classes, then either $E_1 = E_2$ or $E_1 \cap E_2 = \emptyset$.

Proof: If $E_1 = E_2$, then we're done.

Now suppose $E_1 \neq E_2$. Also suppose that $E_1 \cap E_2 \neq \emptyset$ [otherwise we're done].

Since E_1, E_2 are equivalence classes, we know that there are elements $x, y \in S$, such that $E_1 = [x]$, and $E_2 = [y]$.

Since $E_1 \neq E_2$, one of them contains an element not in the other.

(WLOG) Without loss of generality, suppose that } (a)
there is $z \in E_1$ such that $z \notin E_2$.

$\Rightarrow z \in [x] \Rightarrow \underline{x \sim z}$ (1)

and by assumption, $[x] \cap [y] \neq \emptyset$

So, there is some $w \in [x] \cap [y]$, that is

(2) $w \sim x$ & $w \sim y$. (2)

By transitivity and symmetry, (1)+(2)+(3) tell us that

$z \sim x \sim w \sim y \Rightarrow z \sim y$

$\Rightarrow \underline{z \in [y]}$ (b)

(a) and (b) contradict each other. Contradiction!

*** Notation :

(1) We say that $S = \underline{S_1 \amalg S_2}$ if

$$S = S_1 \cup S_2 \quad \text{and also} \quad S_1 \cap S_2 = \emptyset.$$

(2) We often write $S = \underline{\bigcup_{i \in I} S_i}$ or $S = \underline{\bigamalg_{i \in I} S_i}$

to mean a union / disjoint union over an
"index set" I.

*** Proposition : Let \sim be an equivalence relation on S . Let $\{S_i \mid i \in I\}$ be the set of equivalence classes.

$$\text{Then } S = \underline{\bigamalg_{i \in I} S_i}.$$

* Often, we just write I to mean an unspecified index set

In this case, I is enumerating all the equivalence classes.

* Proposition translates as: a set S , with an equivalence relation \sim , is the disjoint union over all of its equivalence classes.

Proof: Let \sim be an equivalence relation on S .

Let $\{S_i \mid i \in I\}$ be the set of equivalence classes.

(1) Consider two ^{distinct} equivalence classes S_i & S_j

Then by the previous result, $S_i \cap S_j = \emptyset$.

(2) Every $x \in S$ is in some equivalence class, namely $[x]$. So $[x] = S_i$ for some i

$$\Rightarrow \underline{\bigcup_{i \in I} S_i} = S.$$

** How to think about equivalence relations / equivalence classes

Heuristic: A relation that identifies some shared property between elements in your set is usually an equivalence relation.

An equivalence class can often be described as "all elements of S that have the same ..."

Example: $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is a multiple of } 17\}$

Equivalence classes consist of all elements of \mathbb{Z} that have the same remainder when divided by 17.

* Modular arithmetic

We'll define a new system of arithmetic
(+, -, *, /, ...) using equivalence classes on \mathbb{Z} .

** Baby example :

Consider $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is even}\}$

This is an equivalence relation. \cdot [modulus = 2]

Equivalence classes :

$$\{x \in \mathbb{Z} \mid x \text{ is even}\} = \{\dots, -2, 0, 2, 4, \dots\}$$

$$\{x \in \mathbb{Z} \mid x \text{ is odd}\} = \{\dots, -3, -1, 1, 3, \dots\}$$

$$\{\text{evens}\} = [0] = [2020] = [42] = [-22]$$

$$\{\text{odds}\} = [2021] = [-5] = [3]$$

Typically, we'll use 0 and 1 as our
standard representatives, but they are not
special in any way.