MATH 2301

** Office hour: Monday 1–2 (in person + Zoom)

* Last time: Equivalence classes

*** Proposition

(1) Let $y \in [x]$. Then $x \in [y]$ and $[x] = [y]$

(2) If E_1 and E_2 are two equivalence classes, then either $E_1 = E_2$ or $E_1 \cap E_2 = \phi$.

Proof

(1) Let $y \in [x] \Rightarrow x \sim y$.

By symmetry, we have $y \sim x$

So, $x \in [y]$.

Let $y \in [x]$. $\Rightarrow x \sim y$

To show that $[x] = [y]$, suppose that $z \in [x]$

$\Rightarrow x \sim z$

By symmetry & transitivity, $y \sim z$

$\Rightarrow z \in [y]$

(Similarly if $z \in [y]$ then $z \in [x]$)

$\Rightarrow [x] = [y]$.

(2) If E_1 and E_2 are two equivalence classes, then either $E_1 = E_2$ or $E_1 \cap E_2 = \phi$.

Proof : If $E_1 = E_2$, then we're done.

Now suppose $E_1 \neq E_2$. Also suppose that $E_1 \cap E_2 \neq \phi$ [otherwise we're done].

Since E_1, E_2 are equivalence classes, we know that there are elements $x, y \in S$, such that $E_1 = [x]$, and $E_2 = [y]$.

Since $E_1 \neq E_2$, one of them contains an element not in the other.

(WLOG) Without loss of generality, suppose that there is $z \in E_1$ such that $z \notin E_2$. $\Bigg\}$ ⓐ

$\Rightarrow z \in [x] \Rightarrow x \sim z$ ①

and by assumption, $[x] \cap [y] \neq \phi$

So, there is some $w \in [x] \cap [y]$, that is

② $w \sim x$ & $w \sim y$. ③

By transitivity and symmetry, ①+②+③ tell us that

$z \sim x \sim w \sim y \Rightarrow z \sim y$

$\Rightarrow z \in [y]$ ⓑ

ⓐ and ⓑ contradict each other. Contradiction!

*** Notation :

(1) We say that $S = S_1 \amalg S_2$ if disjoint union.

$S = S_1 \cup S_2$ and also $S_1 \cap S_2 = \phi$.

(2) We often write $S = \bigcup_{i \in I} S_i$ or $S = \coprod_{i \in I} S_i$

to mean a union / disjoint union over an "index set" I.

*** Proposition : Let $\sim$ be an equivalence relation on S. Let $\{S_i \mid i \in I\}$ be the set of equivalence classes.

Then $S = \coprod_{i \in I} S_i$.

* Often, we just write I to mean an unspecified index set
In this case, I is enumerating all the equivalence classes.

* Proposition translates as: a set S, with an equivalence relation $\sim$, is the disjoint union over all of its equivalence classes.

Proof: Let $\sim$ be an equivalence relation on S.

Let $\{S_i \mid i \in I\}$ be the set of equivalence classes.

(1) Consider two distinct equivalence classes S_i & S_j

Then by the previous result, $S_i \cap S_j = \phi$.

(2) Every $x \in S$ is in some equivalence class, namely $[x]$. So $[x] = S_i$ for some i

$\Rightarrow \bigcup_{i \in I} S_i = S$.

** How to think about equivalence relations / equivalence classes

Heuristic: A relation that identifies some shared property between elements in your set is usually an equivalence relation.

An equivalence class can often be described as "all elements of S that have the same ..."

Example: $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y$ is a multiple of $17\}$

Equivalence classes consist of all elements of $\mathbb{Z}$ that have the same remainder when divided by 17.

* Modular arithmetic

We'll define a new system of arithmetic $(+, -, *, /, \ldots)$ using equivalence classes on $\mathbb{Z}$.

** Baby example:

Consider $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x-y \text{ is even}\}$

This is an equivalence relation. [modulus = 2]

Equivalence classes:

$\{x \in \mathbb{Z} \mid x \text{ is even}\} = \{\ldots, -2, 0, 2, 4, \ldots\}$

$\{x \in \mathbb{Z} \mid x \text{ is odd}\} = \{\ldots, -3, -1, 1, 3, \ldots\}$

$\{\text{evens}\} = [0] = [2020] = [42] = [-22]$

$\{\text{odds}\} = [2021] = [-5] = [3]$

Typically, we'll use 0 and 1 as our standard representatives, but they are not special in any way.