MATH 2301

* Assignment 1 due tomorrow on Gradescope!

* Last time : Modular arithmetic

* Posets (preview)

** Def : A relation R on a set S is called a partial order if it is reflexive, anti-symmetric and transitive.

** Notation : Let R be a partial order on a set S.

① We'll say that S is a poset with respect to R.

② If $(a,b) \in R$, we'll say that $a \preceq b$

If $a \preceq b$ & $a \neq b$, we'll say $a \prec b$ or $a \precneqq b$

If $a \preceq b$ then we can write $b \succeq a$

** Examples [Checks left as an exercise]

① $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\}$

E.g in this case $1 \preceq 3$ because $1 \leq 3$.

② $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x \geq y\}$

In this case, $3 \preceq 1$ because $3 \geq 1$

③ S a fixed set.

$\{(A,B) \in \mathcal{P}(S) \times \mathcal{P}(S) \mid A \subseteq B\}$

E.g. $S = \{1, 2\}$: $\emptyset \preceq \{1\}$, $\emptyset \preceq \{2\}$, $\emptyset \preceq \{1,2\}$

$\{1\} \preceq \{1,2\}$, $\{2\} \preceq \{1,2\}$, $\emptyset \preceq \emptyset$, $\{1\} \preceq \{1\}$, etc.

④ $S = \{$sneezy, sleepy, happy, doc, grumpy, dopey, bashful$\}$

$\{(w_1, w_2) \in S \times S \mid \text{length}(w_1) \leq \text{length}(w_2)$ and w_1 is alphabetically $\leq w_2\}$.

doc $\preceq$ sneezy

sleepy & bashful are unrelated.

bashful is only related to itself

"a divides b"

⑤ $\{(a,b) \in \mathbb{N}_1 \times \mathbb{N}_1 \mid \underbrace{a \text{ is a factor of}}_{a|b} b\}$

$\mathbb{N}_1$: not including zero

$2 \preceq 22$ because $2 | 22$

$3 \preceq 57$ because $3 | 57$

** Total orders

Let $(P, \preceq)$ be a poset. Then $(P, \preceq)$ is a total order if whenever $a, b \in P$, either $a \preceq b$ or $b \preceq a$.

[In this case, P is sometimes called a toset.]

*** Notation:

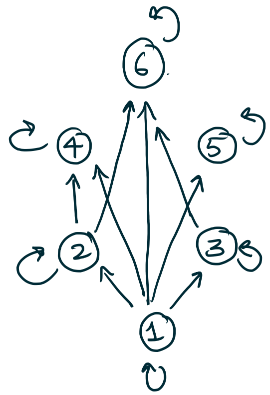
Let a, b be elements of a poset. We say that a & b are comparable if either $a \preceq b$ or $b \preceq a$. Otherwise they are incomparable.

** Hasse diagram

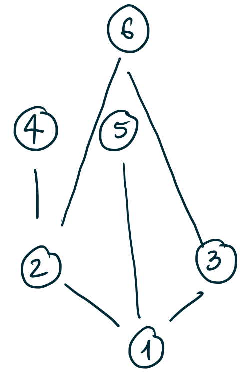
E.g. $S = \{1, 2, 3, 4, 5, 6\}$

$R = \{(a,b) \in S \times S \mid a \text{ is a factor of } b\}$

Graph



Hasse diagram



** Graph → Hasse diagram?

① Delete self-loops

② Delete "shortcut" arrows (anything implied by transitivity)

③ All edges are assumed to be oriented upwards, and so we remove arrowheads.

** Hasse diagram → Graph drawing?

1. Draw arrows (upwards)
2. Fill in transitive arrows
3. Draw self-loops.

** Examples

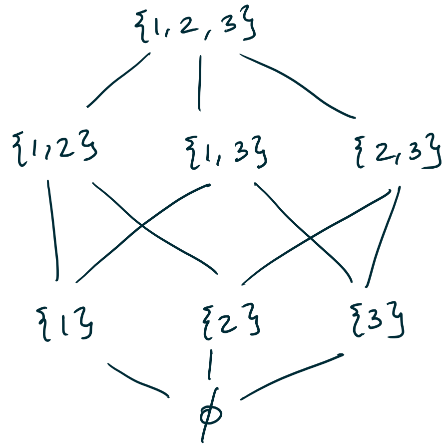
1. $S = \{1,2,3,4\}$

$\{(a,b) \in S \times S \mid a \leq b\}$

The Hasse diagram of a total order is a straight line

2. $S = \{1, 2, 3\}$

$\{(A,B) \in P(S) \times P(S) \mid A \subseteq B\}$



** Topological sort

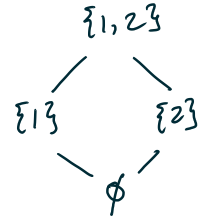
Let $(P, \preceq)$ be a finite poset.

A topological sorting of P is an ordering on all elements of P:

$(p_1, p_2, \cdots, p_n)$, such that:

whenever $p_i \preceq p_j$, we have $i \leq j$.

E.g. Subset poset of $\{1,2\}$: $P = \{\phi, \{1\}, \{2\}, \{1,2\}\}$



Two possible topological sorts:

$(\phi, \{1\}, \{2\}, \{1,2\})$

$(\phi, \{2\}, \{1\}, \{1,2\})$

*** Theorem : Every finite poset has at least one topological sort.

Sketch of Proof : Let $(P, \preceq)$ be a finite poset.

The first element of the ordering should be some $a \in P$ such that for any $b \in P$,

- either $a \preceq b$, or
- a is not comparable with b.

} by finiteness

Once you have the first element, forget about it and do the same procedure to find the second elt...