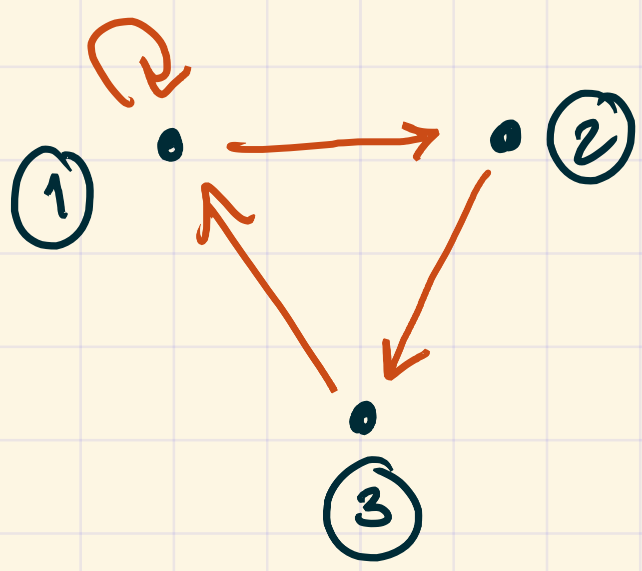
MATH 2301

* <u>Last time</u>: Matrix products + adjacency matrices.

* <u>Today</u>: Paths in graphs & powers of the adjacency matrix.

** <u>Example</u>



$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

** How to understand the $(i,j)^{th}$ entry of A^2 ?

$A^2_{(i,j)}$ = dot product of R_i with C_j

i^{th} row j^{th} column.

<u>E.g.</u> $R_1 = (1, 1, 0)$, $C_1 = (1, 0, 1)$

$$R_1 \cdot C_1 = (1 \cdot 1) + (1 \cdot 0) + (0 \cdot 1)$$

edges ① → ①

edges ① → ①

edges ① → ②

edges ② → ①

edges ① → ③

edges ③ → ①

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \cdot A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = A^3.$$

Proposition: The $(i,j)^{th}$ entry of A^2 counts the number of length 2 paths from i to j in the graph.

Explanation: The k^{th} term in the summation of the dot product gives either 1 or 0:

- it gives a 1 if there is an edge $i \to k$ and an edge $k \to j$
- it gives a zero if at least one of those edges doesn't exist.

** Theorem: The $(i,j)^{th}$ entry of A^k is exactly the number of length-k paths from i to j.

** Explanation : $A^k = A \cdot A^{k-1}$

things that you add up

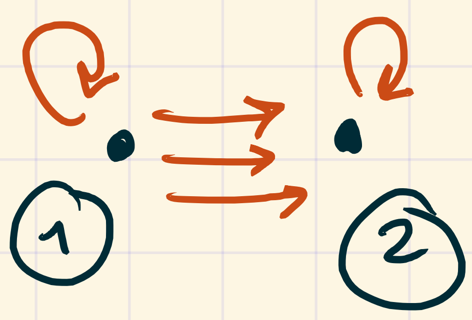
$(A^k)_{(i,j)}$ is a dot product, whose summands are :

$A_{(i,\ell)} \cdot (A^{k-1})_{(\ell,j)}$ ⟵ counts the # length-k paths from i to j, that begin with $i \to \ell \to \cdots \to j$

edges from $i \to j$

by induction, number of length $(k-1)$ paths from $\ell \to j$

** Aside : non-simple graphs



$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$$

** Connectedness of graphs

Let $G = (V, E)$ be a graph. Is there at least one path from the i^{th} vertex to the j^{th} vertex?

(Not necessarily ... but how do we know?)

[Step zero $I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{bmatrix}$ ← length zero paths]

Start with A ← length one paths

A^2 ← length two paths

A^3

$\vdots$

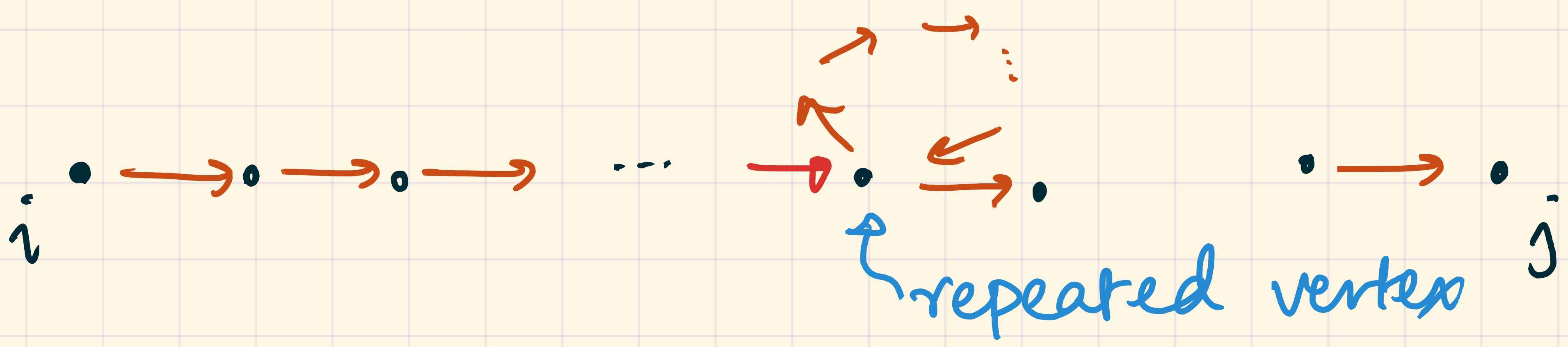
A^k ← length k paths

If there is a path from i to j of some length, it will show up as the $(i, j)^{th}$ entry of the corresponding power.

But what if there isn't a path from i to j?

*** Follow-up question. Suppose G has n vertices. How far would we need to go?

Suppose there is a path from i to j



If the length of this path is $> n$, then at least one vertex is repeated.

So there is some loop within the path.

Just erase that loop to make the path shorter!

Conclusion: If there is at least one path from i to j, the shortest such path can't be too long.

* If length-0 paths are allowed as connections, then any path of length $> n-1$ can be shortened. (because it will repeat an intermediate vertex)

* If length 0 paths are not allowed as connections, then any path of length $> n$ can be shortened (because it will repeat an intermediate vertex.)