

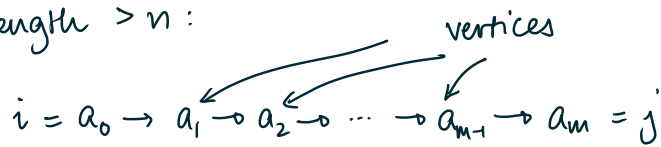
MATH 2301

* Continued from last time :

** Theorem : Let $G = (V, E)$ be a graph with n vertices.

Let $i, j \in V$. Then if there is a ^(non-zero) path from i to j , the shortest (non-zero) path has length $\leq n$.

** Explanation : Suppose we had a path of length $> n$:



Since this path has length $m > n$, at least two of the vertices are equal: $a_k = a_l$ for some $k < l$.

Then we can simply delete the portion of the path from a_k to a_l , to make it shorter.

Continue until your path has length $\leq n$.

Done!

□

** Upshot

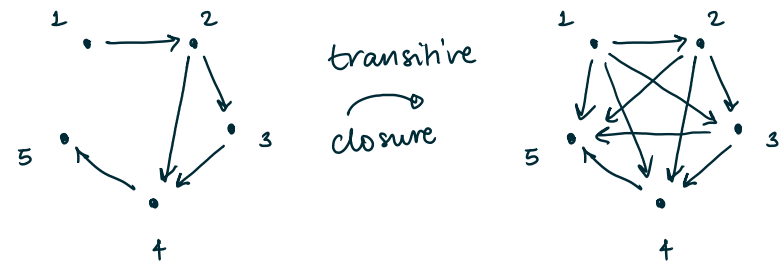
Let $G = (V, E)$ have n vertices.

To find if there is any path from i to j , (of non-zero length), we simply compute

$$(A + A^2 + A^3 + \dots + A^n)_{(i,j)}$$

If this entry is zero, then there is no path from i to j , otherwise this entry gives you the number of paths $i \rightarrow \dots \rightarrow j$ of length $\leq n$.

** Transitive closure



The transitive closure of a relation R is the minimal transitive relation R' such that $R \subseteq R'$ (see worksheet for more)

** Transitive closure using adjacency matrices

Whenever $(A^k)_{(i,j)} \neq 0$, change the (i,j) th entry of A to 1 (we'll elaborate soon).

** Boolean arithmetic

Defined on the set $\{0, 1\}$ with "FALSE" for 0 and "TRUE" for 1.

We have operations

- \vee ("OR") : $0 \vee 0 = 0$
 $0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$

- \wedge ("AND") : $0 \wedge 0 = 0$
 $0 \wedge 1 = 0$ $1 \wedge 1 = 1$
 $1 \wedge 0 = 0$

* Not the same as addition mod 2!

*** Boolean matrix sum/product

- Defined for matrices with entries 0 & 1.

Same procedure as usual matrix sum/product,

but anytime you

- add numbers, you "OR" them
 $a + b \rightsquigarrow a \vee b$

- multiply numbers, you "AND" them
 $a \cdot b \rightsquigarrow a \wedge b$

[$\{0, 1\}$, together with \vee , \wedge , forms a "semiring"]

*** Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Notation: $A * B =$ Boolean product of A with B .

$$A * A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) \end{bmatrix}$$

$$A^{*2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

On the other hand, $A \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq A * A$.

[Boolean arithmetic can't count, but it tells you yes/no]

* The entries of A^{*k} , if A is an adjacency matrix, tell you whether there are paths of length exactly k .

** Theorem: The adjacency matrix of the transitive closure of a relation with adjacency matrix A is given by

$$(A \vee A^{*2} \vee A^{*3} \vee \dots \vee A^{*n})$$

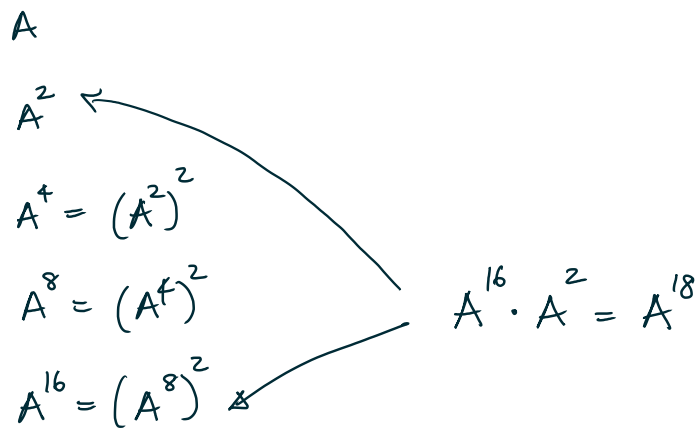
↑
Boolean matrix sum.

** Repeated squaring.

Example: How to compute A^{18} ?

Traditionally, $\underbrace{A \cdot A \cdot \dots \cdot A}_{17 \text{ matrix products}}$.

Better:



4 operations + 1 final operation = 5.

$$A^{37} = A^{32} \cdot A^4 \cdot A^1$$

Secretly, binary arithmetic / base-2 writing.

Let $n \geq 0$ be an integer. Writing n in base-2 means writing it as the sum of distinct powers of 2.

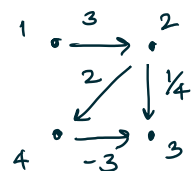
In practice, how to compute this?

$$57 = 2^5 + 25 = 2^5 + 2^4 + 2^3 + 1$$

[Successively subtract the largest power of 2 that is \leq your number.]

(you'll see this again!)

** Weighted adjacency matrices



Consider a graph $G=(V, E)$ together with a weight on each edge.

Each weight can be any real number.

Weighted adjacency matrix:

?	3	?	?
?	?	1/4	2
?	?	?	?
?	?	-3	?

⓪ entries are those where there is no edge, and we'll decide later what goes here.