MATH 2301

* Continued from last time :

** **Theorem** : Let $G = (V, E)$ be a graph with n vertices.
Let $i, j \in V$. Then if there is a (non-zero) path from i to j, the shortest (non-zero) path has length $\leq n$.

** **Explanation** : Suppose we had a path of length $> n$:

vertices

$$i = a_0 \to a_1 \to a_2 \to \cdots \to a_{m-1} \to a_m = j$$

Since this path has length $m > n$, at least two of the vertices are equal : $a_k = a_\ell$ for some $k < \ell$.
Then we can simply delete the portion of the path from a_k to a_ℓ, to make it shorter.

Continue until your path has length $\leq n$.
Done ! $\square$

** **Upshot**

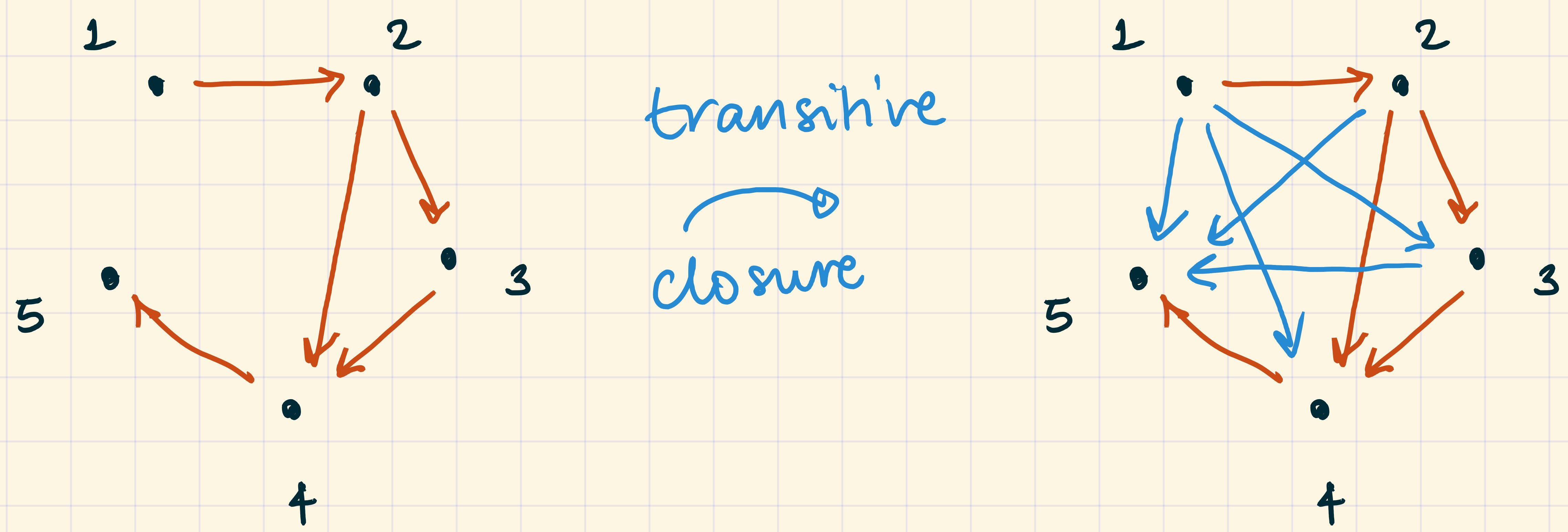
Let $G = (V, E)$ have n vertices.

To find if there is any path from i to j, (of nonzero length), we simply compute

$$\left(A + A^2 + A^3 + \cdots + A^n\right)_{(i,j)}$$

If this entry is zero, then there is no path from i to j, otherwise this entry gives you the number of paths $i \to \cdots \to j$ of length $\leq n$.

** **Transitive closure**



The transitive closure of a relation R is the minimal transitive relation R' such that $R \subseteq R'$

(see worksheet for more)

** **Transitive closure using adjacency matrices**

Whenever $(A^k)_{(i,j)} \neq 0$, change the $(i,j)^{th}$ entry of A to 1 (we'll elaborate soon).

** Boolean arithmetic

Defined on the set $\{0, 1\}$ ("FALSE" for 0, "TRUE" for 1)

We have operations

- $\vee$ ("OR"): $0 \vee 0 = 0$
  $0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$

- $\wedge$ ("AND"): $0 \wedge 0 = 0$
  $0 \wedge 1 = 0$ $1 \wedge 1 = 1$
  $1 \wedge 0 = 0$

* Not the same as addition mod 2!

*** Boolean matrix sum/product

- Defined for matrices with entries 0 & 1.

Same procedure as usual matrix sum/product, but anytime you

- add numbers, you "OR" them

  $a + b \rightsquigarrow a \vee b$

- multiply numbers, you "AND" them

  $a \cdot b \rightsquigarrow a \wedge b$

[$\{0,1\}$, together with $\vee$, $\wedge$, forms a "semiring".]

*** Example

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Notation: $A * B$ = Boolean product of A with B.

$$A * A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 0) \vee (1 \wedge 1) \end{bmatrix}$$

$$A^{*2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

On the other hand, $A \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq A * A$.

[Boolean arithmetic can't count, but it tells you yes/no]

* The entries of A^{*k}, if A is an adjacency matrix, tell you whether there are paths of length exactly k.

** Theorem: The adjacency matrix of the transitive closure of a relation with adjacency matrix A is given by

$$(A \vee A^{*2} \vee A^{*3} \vee \cdots \vee A^{*n}).$$

Boolean matrix sum.

** Repeated squaring.

Example : How to compute A^{18} ?

Traditionally, $\underbrace{A \cdot A \cdot \cdots \cdot A}$ 17 matrix products.

Better :

A

A^2

$A^4 = (A^2)^2$

$A^8 = (A^4)^2$ $\quad\quad A^{16} \cdot A^2 = A^{18}$

$A^{16} = (A^8)^2$

4 operations + 1 final operation = 5.

$$A^{37} = A^{32} \cdot A^4 \cdot A^1$$

Secretly, binary arithmetic / base-2 writing.

Let $n \geq 0$ be an integer. Writing n in base-2 means writing it as the sum of distinct powers of 2.

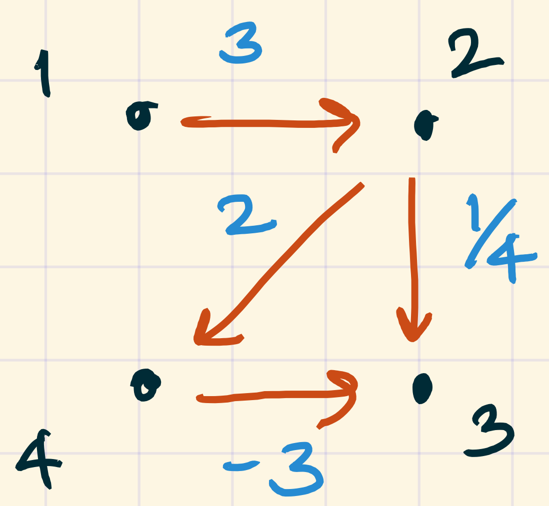
In practice, how to compute this?

$$57 = 2^5 + 25 = 2^5 + 2^4 + 2^3 + 1$$

[Successively subtract the largest power of 2 that is $\leq$ your number.]

(you'll see this again!)

** Weighted adjacency matrices



Consider a graph $G = (V, E)$ together with a weight on each edge.

Each weight can be any real number.

Weighted adjacency matrix:

$$\begin{bmatrix} ? & 3 & ? & ? \\ ? & ? & \frac{1}{4} & 2 \\ ? & ? & ? & ? \\ ? & ? & -3 & ? \end{bmatrix}$$

? entries are those where there is no edge, and we'll decide later what goes here.