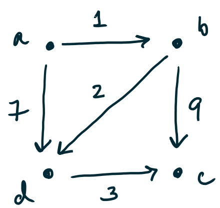
MATH 2301

* Path vs walk : A walk is a sequence of vertices in a graph : $(v_0, v_1, \dots, v_n)$, such that any successive pair (v_i, v_{i+1}) is an edge.
  In many sources, a path is defined as a walk where no vertices repeat.

  BUT we'll use path & walk interchangeably.

* Continued from last time : Weighted adjacency matrices

** Example



Most commonly, we think of an edge weight as a "cost" for each edge, so we're usually looking for the least cost to travel, say.

$$W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

"∞" is just a symbol used to denote a cost that is extremely high.

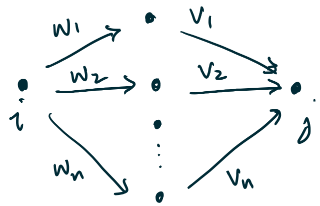
We'll regard ∞ as being $>$ any real number.

** Lowest-cost paths

Consider a weighted graph whose edge weights are non-negative.

Q: How can we find the lowest-cost path from i to j?

** min-plus product

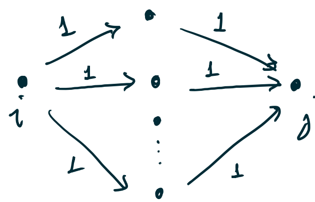


To find the lowest-cost of a 2-step path from i to j, we first sum up the various possibilities:

$w_1+v_1,\ w_2+v_2,\ w_3+v_3,\ \dots,\ w_n+v_n$, and then we take the minimum

$$\min\{w_1+v_1, w_2+v_2, \dots, w_n+v_n\}$$

** Comparison w/ the usual adjacency matrix square:



We multiplied the various possibilities together:

$1\cdot 1,\ 1\cdot 1, \dots, 1\cdot 1$, and then added them together to give the number of length-2 paths

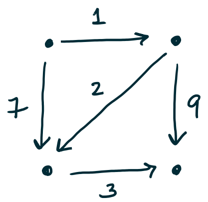
$\rightarrow (1\cdot 1)+(1\cdot 1)+\dots+(1\cdot 1)$.

** **Defn**: The <u>min-plus product</u> of two matrices

is just like a usual matrix product, but anytime we were

- multiplying numbers ⇝ we now add them
- adding numbers ⇝ we take the minimum.

** **Example**



$$W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

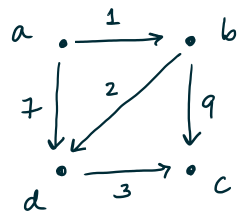
Notation: $A \odot B$ = min-plus product

$$W \odot W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix} \odot \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix}\min(\infty+\infty,\ 1+\infty, \\ \infty+\infty,\ 7+\infty)\end{matrix} & \begin{matrix}\min(\infty+1,\ 1+\infty, \\ \infty+\infty,\ 7+\infty)\end{matrix} & \begin{matrix}\min(\infty+\infty,\ 1+9, \\ \infty+\infty,\ 7+3)\end{matrix} & \begin{matrix}\min(\infty+7,\ 1+2, \\ \infty+\infty,\ 7+\infty)\end{matrix} \\ = \infty & = \infty & = 10 & = 3 \\ \infty & \infty & 5 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

$$W \odot W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix} \odot \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

$$= \begin{bmatrix} \infty & \infty & 10 & 3 \\ \infty & \infty & 5 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$



$$W \odot W \odot W = W^{\odot 3} = \begin{bmatrix} \infty & \infty & 6 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

Exercise: Check $W^{\odot 3}$ using min-plus product

Theorem: Let G be a weighted graph with n vertices, and W its weighted adjacency matrix.

Then the matrix that records lowest-cost paths of any length between vertices is:

$$W \ \underset{\sim}{\min}\ W^{\odot 2} \ \min\ W^{\odot 3} \cdots \ \min\ W^{\odot n}$$

↑ take entry-wise minimum.

* Reminder : functions

** Defn : A function $f: A \to B$ is a relation $f \subseteq A \times B$, such that: for each $a \in A$ there is a unique $b \in B$ such that $(a,b) \in f$.

In this case, we write $f(a) = b$.

Examples :

① $f : P(\{1,2,3\}) \to \mathbb{N}$ defined as
$f(A) =$ number of elements in A.
$f(\{1,2\}) = 2$, $f(\phi) = 0$, $f(\{3\}) = 1$ etc

② $f : P(\{1,2,3\}) \to P(\{1,2,3\})$ defined as

$f(A) = A \cup \{1,2\}$

$f(\{1,2\}) = \{1,2\}$, $f(\phi) = \{1,2\}$, $f(\{3\}) = \{1,2,3\}$ etc.

*** Exercise : How many different functions are there from $\{1,2,3\}$ to $\{a,b\}$?

a or b   a or b   a or b

$2 \times 2 \times 2 = 8$