MATH 2301

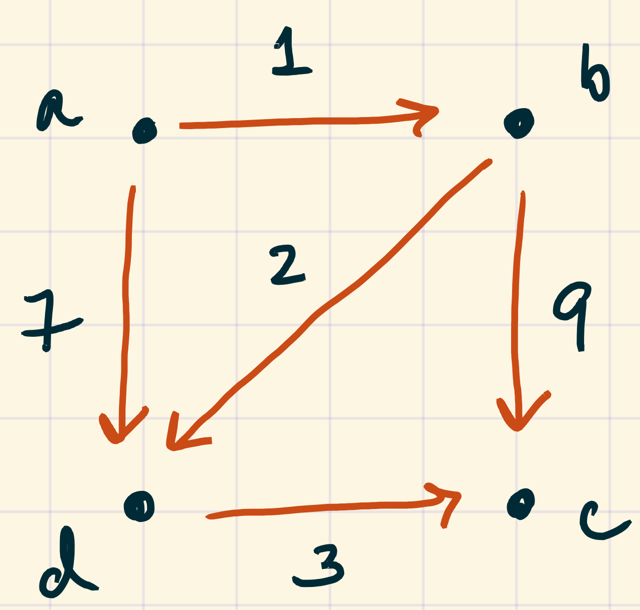
* Path vs walk : A walk is a sequence of vertices in a graph : $(v_0, v_1, \ldots, v_n)$, such that any successive pair (v_i, v_{i+1}) is an edge.

  In many sources, a path is defined as a walk where no vertices repeat.

  BUT we'll use path & walk interchangeably.

* Continued from last time : Weighted adjacency matrices

** Example



Most commonly, we think of an edge weight as a "cost" for each edge, so we're usually looking for the least cost to travel, say.

$$W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

"∞" is just a symbol used to denote a cost that is extremely high.

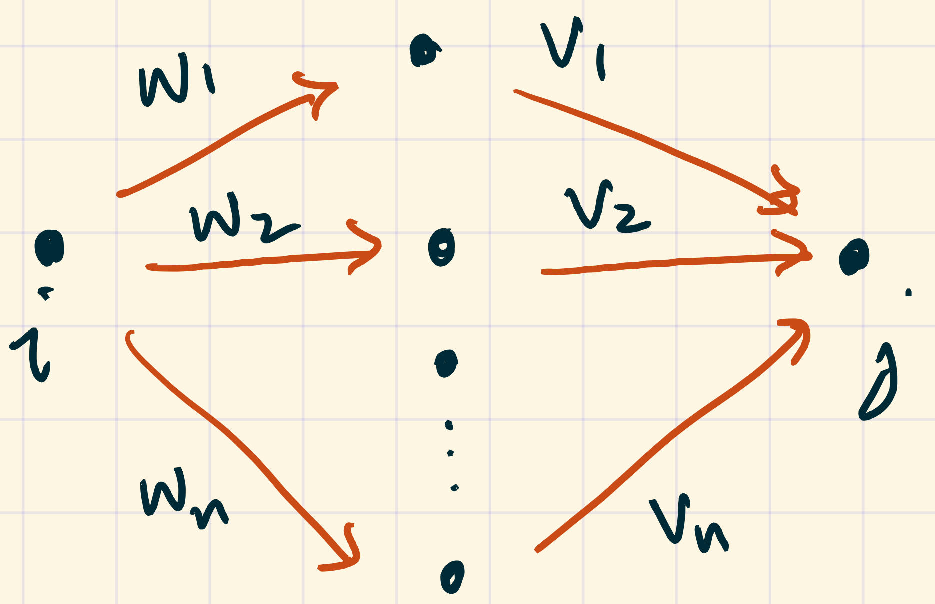
We'll regard ∞ as being $>$ any real number.

** Lowest-cost paths

Consider a weighted graph whose edge weights are non-negative.

Q: How can we find the lowest-cost path from i to j?

** min-plus product

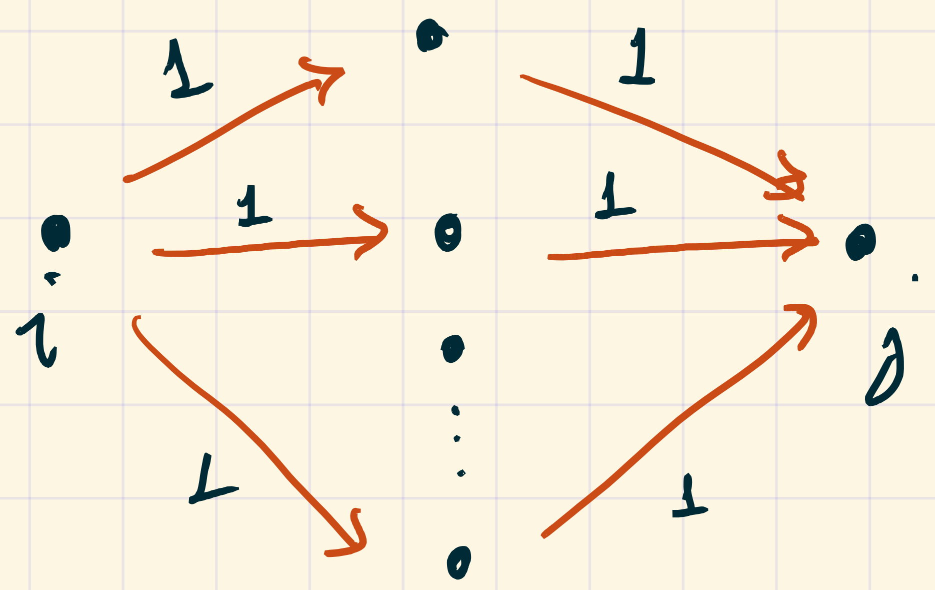


To find the lowest-cost of a 2-step path from i to j, we first sum up the various possibilities:

w_1+v_1, w_2+v_2, w_3+v_3, $\dots$, w_n+v_n, and then we take the minimum

$$\min\{w_1+v_1, w_2+v_2, \dots, w_n+v_n\}$$

** Comparison w/ the usual adjacency matrix square:



We multiplied the various possibilities together:

$1\cdot 1$, $1\cdot 1$, $\dots$, $1\cdot 1$, and then added them together to give the number of length-2 paths

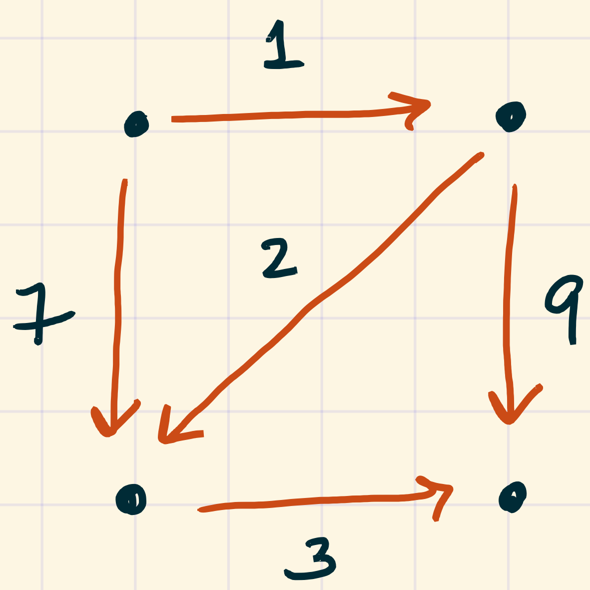
$\rightarrow$ $(1\cdot 1)+(1\cdot 1)+\dots+(1\cdot 1)$.

** <u>Defn</u> : The <u>min-plus product</u> of two matrices

is just like a usual matrix product, but anytime we were

- multiplying numbers ⇝ we now add them
- adding numbers ⇝ we take the minimum.

** <u>Example</u>



$$W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

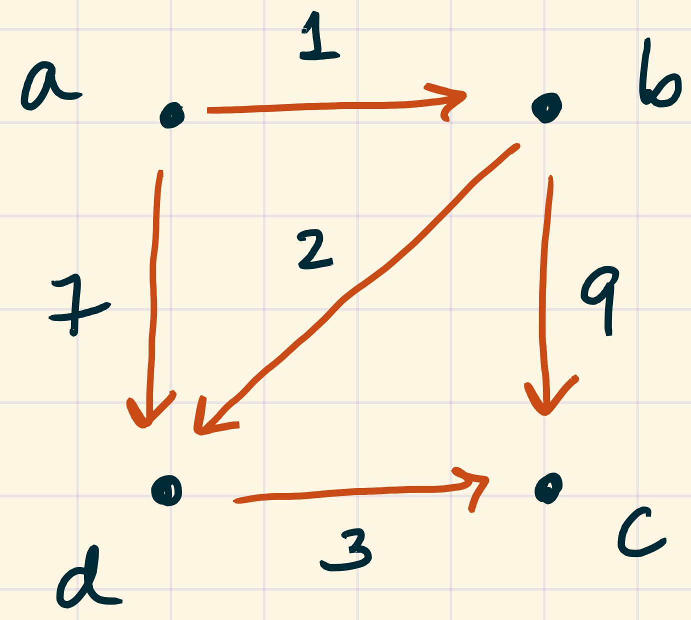
<u>Notation</u> : $A \odot B$ = min-plus product

$$W \odot W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix} \odot \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{\min(\infty+\infty, 1+\infty, \infty+\infty, 7+\infty)}_{=\infty} & \underbrace{\min(\infty+1, 1+\infty, \infty+\infty, 7+\infty)}_{=\infty} & \underbrace{\min(\infty+\infty, 1+9, \infty+\infty, 7+3)}_{=10} & \underbrace{\min(\infty+7, 1+2, \infty+\infty, 7+\infty)}_{=3} \\ \infty & \infty & 5 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

$$W \odot W = \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix} \odot \begin{bmatrix} \infty & 1 & \infty & 7 \\ \infty & \infty & 9 & 2 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty \end{bmatrix}$$

$$= \begin{bmatrix} \infty & \infty & 10 & 3 \\ \infty & \infty & 5 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$



$$W \odot W \odot W = W^{\odot 3} = \begin{bmatrix} \infty & \infty & 6 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

Exercise: Check $W^{\odot 3}$ using min-plus product

Theorem: Let G be a weighted graph with n vertices, and W its weighted adjacency matrix.

Then the matrix that records lowest-cost paths of any length between vertices is:

$$W \ \min\ W^{\odot 2} \ \min\ W^{\odot 3} \cdots \ \min\ W^{\odot n}$$

take entry-wise minimum.

* Reminder: functions

** Defn: A function $f: A \to B$ is a relation $f \subseteq A \times B$, such that: for each $a \in A$ there is a unique $b \in B$ such that $(a,b) \in f$.

In this case, we write $f(a) = b$.

Examples:

① $f: \mathcal{P}(\{1,2,3\}) \to \mathbb{N}$ defined as

$f(A)$ = number of elements in A.

$f(\{1,2\}) = 2$, $f(\phi) = 0$, $f(\{3\}) = 1$ etc

② $f: \mathcal{P}(\{1,2,3\}) \to \mathcal{P}(\{1,2,3\})$ defined as

$f(A) = A \cup \{1,2\}$

$f(\{1,2\}) = \{1,2\}$, $f(\phi) = \{1,2\}$, $f(\{3\}) = \{1,2,3\}$ etc.

*** Exercise: How many different functions are there from $\{1,2,3\}$ to $\{a,b\}$?

a or b   a or b   a or b

$2 \times 2 \times 2 = 8$