MATH 2301

* Clarification about diagonal entries of weighted adjacency matrix

** Method 1 [If we are only looking for paths of non-zero length]

- Set diagonal entries of W to be either
  - the weight of the self-loop at that vertex if a self-loop is present, or
  - ∞ if there is no self-loop at that vertex.
- To find all min-cost paths, compute

$$W \text{ min } W^{\odot 2} \text{ min } \cdots \text{ min } W^{\odot n}$$

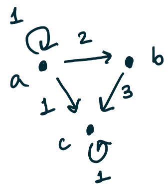
** Method 2 : [If zero-length paths are allowed]

- Set all diagonal entries of W to be zero.
- To find all min-cost paths, simply compute $W^{\odot n}$

* Note that in this case, $W^{\odot k}$ will give min-cost paths of length $\leq k$.

MAKE THIS THE DEFAULT

** Example



Method 1

$$W = \begin{bmatrix} 1 & 2 & 1 \\ \infty & \infty & 3 \\ \infty & \infty & 1 \end{bmatrix}$$

$$W^{\odot 2} = \begin{bmatrix} 2 & 3 & 2 \\ \infty & \infty & 4 \\ \infty & \infty & 2 \end{bmatrix}$$

min-cost paths of length = 2

$$W^{\odot 3} = \begin{bmatrix} 3 & 4 & 3 \\ \infty & \infty & 5 \\ \infty & \infty & 3 \end{bmatrix}$$

Method 2

$$W = \begin{bmatrix} 0 & 2 & 1 \\ \infty & 0 & 3 \\ \infty & \infty & 0 \end{bmatrix}$$

$$W^{\odot 2} = \begin{bmatrix} 0 & 2 & 1 \\ \infty & 0 & 3 \\ \infty & \infty & 0 \end{bmatrix}$$

min-cost of all paths of length ≥ 0 and ≤ 2

$$W^{\odot 3} = \begin{bmatrix} 0 & 2 & 1 \\ \infty & 0 & 3 \\ \infty & \infty & 0 \end{bmatrix}$$

W min $W^{\odot 2}$ min $W^{\odot 3}$ (method 1)

$$= \begin{bmatrix} 1 & 2 & 1 \\ \infty & \infty & 3 \\ \infty & \infty & 1 \end{bmatrix}$$

method 2: $W^{\odot 3}$ =

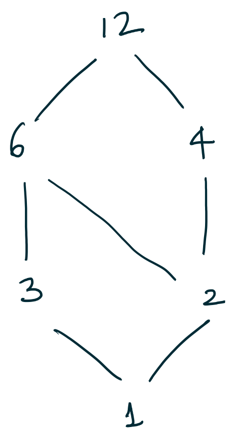
$$\begin{bmatrix} 0 & 2 & 1 \\ \infty & 0 & 3 \\ \infty & \infty & 0 \end{bmatrix}$$

* The resulting matrices are equal, except possibly on the diagonal. We'll prefer method 2.

* Algebra on posets

Example $\{ (a,b) \in \mathbb{N}_1 \times \mathbb{N}_1 \mid a \leq 12, b \leq 12,$
$a \mid 12,\ b \mid 12$ and $a \mid b \}$

↳ a is a factor of b.



** **Definition**: Let $(P, \preceq)$ be any poset.

An interval $[x, y]$ is the set

$$\{ z \in P \mid x \preceq z \text{ and } z \preceq y \}$$

Example: ① $[1, 6]$ in the above poset is:

$\{1, 2, 3, 6\}$

② $[2, 2] = \{2\}$

③ $[2, 3] = \phi$

④ $[6, 1] = \phi$.

⑤ $[6, 12] = \{6, 12\}$

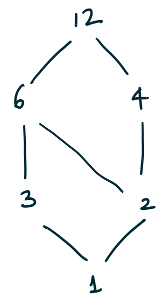
⑥ $[3, 12] = \{3, 6, 12\}$

** The incidence algebra

Let $(P, \preceq)$ be a finite poset.

Let $I(P)$ be the set of all non-empty intervals in P.

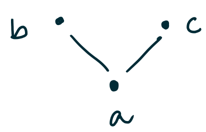
Example (previous example)



$I(P) = \{ [1,1], [2,2], [3,3], [4,4], [6,6],$
$[12,12], [1,2], [1,3], [2,6], [3,6], [2,4],$
$[6,12], [4,12], [1,4], [3,12],$
$[1,6], [2,12], [1,12] \}$

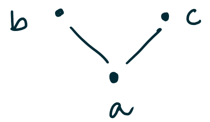
** **Defn**: The incidence algebra of P is the set of all functions from $I(P)$ to $\mathbb{R}$.
It is denoted as $\mathcal{A}(P)$.

*** **Example**



$I(P) = \{ [a,a], [b,b], [c,c],$
$[a,b], [a,c] \}$

$$\mathcal{A}(P) = \{ f : I(P) \to \mathbb{R} \}$$



Examples

① $f([x,y]) = 0$ for any $[x,y] \in I(P)$

② $f([x,y]) =$ number of elements in $[x,y]$

$$f([a,a]) = 1 = f([b,b]) = f([c,c])$$

$$f([a,b]) = 2 = f([a,c])$$

③ $f([x,y]) = 1$ for any $[x,y]$

④ $$f([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

** What kind of object is $\mathcal{A}(P)$?

- How many elements are in it?

  If P has at least one element, then infinitely many!

- What can we do with them?

*** Addition

Suppose $f, g \in \mathcal{A}(P)$. Then you can construct "$(f+g)$" $\in \mathcal{A}(P)$, defined as:

$$(f+g)([x,y]) = f([x,y]) + g([x,y])$$

new element of $\mathcal{A}(P)$

*** Scalar multiplication

Let $f \in \mathcal{A}(P)$ and let $r \in \mathbb{R}$.

Define $(r \cdot f) \in \mathcal{A}(P)$ as:

$$(r \cdot f)([x,y]) := r \cdot f([x,y])$$