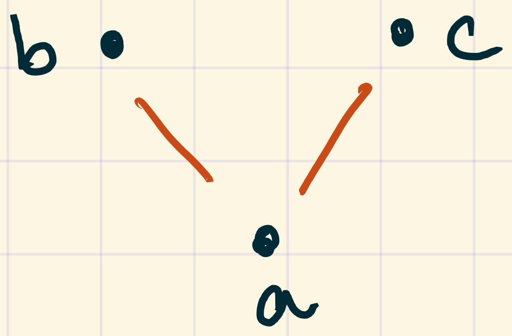
MATH 2301

* Continued: incidence algebra.



$I(P) =$ set of non-empty intervals

$\{[a,a], [b,b], [c,c], [a,b], [a,c]\}$

$$\mathcal{A}(P) = \{f : I(P) \to \mathbb{R}\}$$

** Addition on $\mathcal{A}(P)$

Given $f, g \in \mathcal{A}(P)$, we defined

$(f+g) \in \mathcal{A}(P)$ as

$$(f+g)([x,y]) := f([x,y]) + g([x,y])$$

Example: $f([x,y]) = 1$ for all x, y

$(f+f)([x,y]) = 2$ for all x, y.

$\quad\quad ||$

$(2f)([x,y]) = 2$ for all x, y

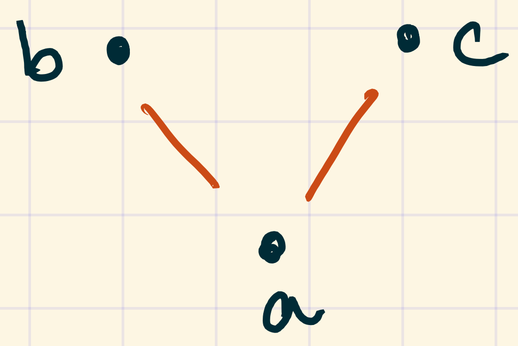
** Scalar multiplication

If $f \in \mathcal{A}(P)$ and $r \in \mathbb{R}$, we defined

$(r \cdot f) \in \mathcal{A}(P)$ as

$$(rf)([x,y]) := r \cdot f([x,y])$$

Example : $f([x,y])$ = number of elements in $[x,y]$



$f([a,b]) = 2$

$(3.5\, f)([a,c]) = 7$

$(3.5 f)([a,a]) = 3.5$

** Some special named functions

Let $(P, \preceq)$ be any poset

*** "Zeta" $\quad \zeta \in \mathcal{A}(P)$

$$\zeta([x,y]) = 1 \quad \text{for any } [x,y]$$

*** "Delta" $\quad \delta \in \mathcal{A}(P)$

$$\delta([x,y]) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

** Convolution product

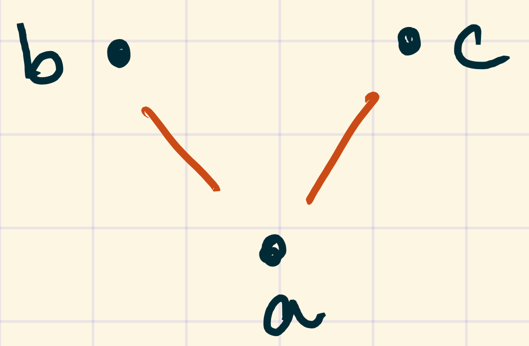
Let $f \in A(P)$ and $g \in A(P)$.

<u>Def</u> : The convolution product of f & g is an element of $A(P)$, denoted $f * g$, defined as:

$$(f*g)([x,y]) = \sum_{x \leq z \leq y} f([x,z]) \cdot g([z,y])$$

*<u>Note</u>: The order of multiplication is important!

** Examples



$$\zeta([x,y]) = 1$$

$$\delta([x,y]) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{else} \end{cases}$$

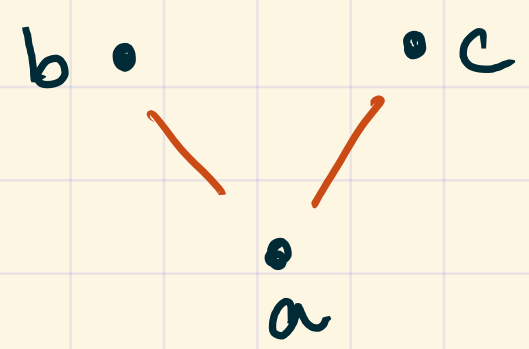
- $(\zeta * \zeta)([a,b]) = \sum_{a \leq z \leq b} \zeta([a,z]) \cdot \zeta([z,b])$

$= \zeta([a,a]) \cdot \zeta([a,b]) + \zeta([a,b]) \cdot \zeta([b,b])$

$= 1 \cdot 1 + 1 \cdot 1 = 2$

- $(\zeta * \zeta)([a,a]) = \sum_{a \leq z \leq a} \zeta([a,z]) \cdot \zeta([z,a])$

$= \zeta([a,a]) \cdot \zeta([a,a]) = 1.$



$$(\zeta * \delta)([a,c])$$

$$= \sum_{a \leq z \leq c}' \zeta([a,z])\, \delta([z,c])$$

$$= \underbrace{\zeta([a,a])}_{=1}\, \underbrace{\delta([a,c])}_{=0} + \underbrace{\zeta([a,c])}_{=1}\, \underbrace{\delta([c,c])}_{=1}$$

$$= 1$$

$$(\zeta * \delta)([b,b]) = 1$$

Let P be any finite poset.

Let $f \in \mathcal{A}(P)$.

$$(f * \delta)([x,y]) = \sum_{x \leq z \leq y}' f([x,z]) \cdot \underbrace{\delta([z,y])}_{\substack{=0 \text{ unless} \\ z=y}}$$

$$= f([x,y]) \cdot \delta([y,y])$$

$$(f * \delta)([x,y]) = f([x,y])$$

$$(\delta * f)([x,y]) = \sum_{x \leq z \leq y}' \underbrace{\delta([x,z])}_{\substack{=0 \\ \text{unless } z=x}} \cdot f([z,y])$$

$$= \delta([x,x]) \cdot f([x,y]) = f([x,y])$$

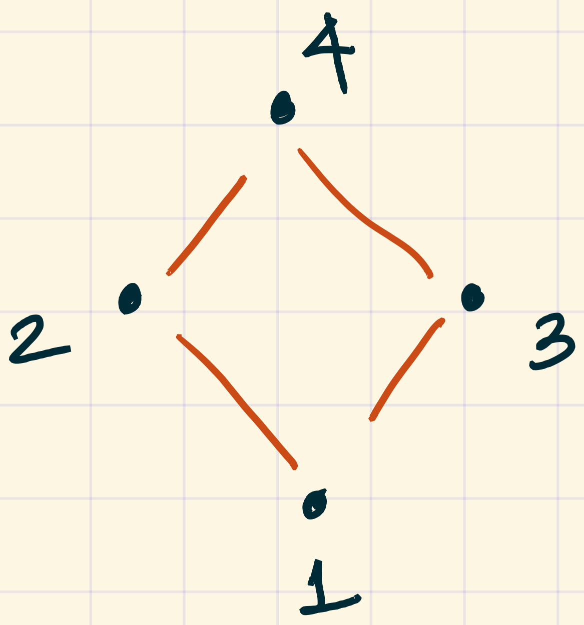
$$(\delta * f)([x,y]) = f([x,y])$$

** **Theorem** : Let $(P, \preceq)$ be any finite poset. The element $\delta \in \mathcal{A}(P)$ is the multiplicative identity for convolution, i.e : if $f \in \mathcal{A}(P)$, we have

$$f * \delta = f \quad \text{and}$$

$$\delta * f = f .$$

Proof : Above calculation.



$$f([x,y]) = (y-x)$$

$$g([x,y]) = \frac{y}{x}$$

$$(f*g)([1,4]) = \sum_{1 \preceq z \preceq 4} f([1,z]) \cdot g([z,4])$$

$$= \sum_{1 \preceq z \preceq 4} (z-1)\frac{4}{z} = (1-1)\left(\frac{4}{1}\right) \quad \rightsquigarrow z=1$$

$$+ (2-1)\left(\frac{4}{2}\right) \quad \rightsquigarrow z=2$$

$$+ (3-1)\left(\frac{4}{3}\right) \quad \rightsquigarrow z=3$$

$$+ (4-1)\left(\frac{4}{4}\right) \quad \rightsquigarrow z=4$$

$$= 2 + \frac{8}{3} + 3 = 5 + \frac{8}{3} = (f*g)([1,4])$$

$$(g*f)([1,4]) = \sum_{1 \leq z \leq 4} \binom{z}{1} \cdot (4-z)$$

$$= \sum_{1 \leq z \leq 4} z(4-z)$$

$$= 1(4-1) + 2(4-2) + 3(4-3) + 4(4-4)$$

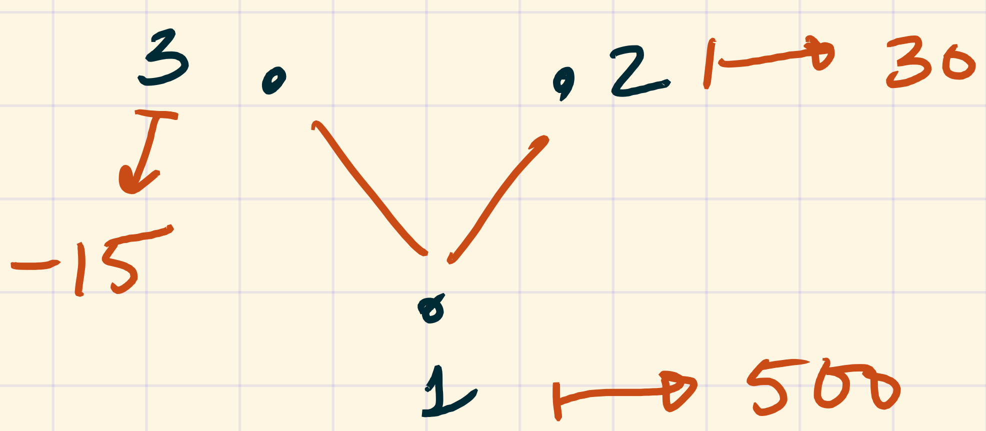
$$= 3 + 4 + 3 = 10. = (g*f)([1,4]).$$

** Functions on posets and one-sided convolution.

A function on a poset P is simply a function

$$p : P \to \mathbb{R}.$$

Example



$$p : P \to \mathbb{R}$$

$p(1) = 500$
$p(2) = 30$
$p(3) = -15$

Let P be a poset.

Let $\underbrace{f \in \mathcal{A}(P)}_{\text{elt of incidence algebra}}$ and $\underbrace{p : P \to \mathbb{R}}_{\text{a function on poset}}$

* <u>One-sided convolution</u>

Want to produce $(f * p)$, a new function on the poset P.

$$(f*p)(x) = \sum_{x \leq z} f([x,z]) \cdot p(z)$$

arbitrary element of P

Similarly, on the other side:

$$(p*f)(x) = \sum_{z \leq x} p(z) f([z,x])$$