MATH 2301

* Mid-semester exam next week

* Convolution reminders

** If $f, g \in \mathcal{A}(P)$, then $f * g \in \mathcal{A}(P)$

$$(f*g)([x,y]) = \sum_{x \leq z \leq y} f([x,z]) \cdot g([z,x])$$

Rmks

In general,

- $(f*g)([x,y]) \neq f([x,y]) \cdot g([x,y])$

- $(f*g)([x,y]) \neq (g*f)([x,y])$ (not commutative)

- $((f*g)*h) = (f*(g*h))$ (it is associative)

** One-sided convolution

Let $f \in \mathcal{A}(P)$, $p : P \to \mathbb{R}$.

Then $f*p$, $p*f$ are functions $P \to \mathbb{R}$.

- $(f*p)(x) = \sum_{x \leq z} f([x,z]) \cdot p(z)$

- $(p*f)(x) = \sum_{z \leq x} p(z) \cdot f([z,x])$

* Invertibility (with respect to $*$)

** Def : An element $f \in \mathcal{A}(P)$ is said to be invertible if there exists $g \in \mathcal{A}(P)$, such that $f * g = \delta$

$$\left[ \delta([x,y]) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} \right]$$ "Kronecker delta function"

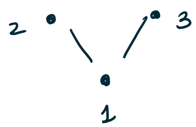
** Rmks

(1) In that case, we say that $g = f^{-1}$ ← inverse in the context of $*$, not in the context of inverting functions.

(2) $\delta$ is its own inverse : $\delta * \delta = \delta$

(3) In this case, it is also true that $g * f = \delta$.

** Important example : $\zeta$.




Q : Is $\zeta$ invertible ?

$\zeta([x,y]) = 1 \quad \forall \; x \leq y$

Let's try to find an inverse. Suppose an inverse exists. Call it $\zeta^{-1}$.

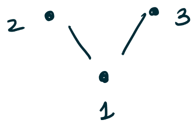
We need :

$$(\zeta * \zeta^{-1}) = \delta$$

This means :

$$(\zeta * \zeta^{-1})([x,y]) = \delta([x,y]) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

i.e $$\sum_{x \le z \le y} \underset{\overset{\shortparallel}{1}}{\zeta([x,z])} \cdot \zeta^{-1}([z,y]) = \delta([x,y])$$



$$\sum_{x \le z \le y} \zeta^{-1}([z,y]) = \delta([x,y])$$

Let's try a particular interval, say $[2,2]$
i.e. $x = 2$, $y = 2$

$$\zeta^{-1}([2,2]) = \delta([2,2]) = 1$$

$\Rightarrow$ $\zeta^{-1}([2,2])$ must be equal to $1$.

$\zeta^{-1}([1,1])$ and $\zeta^{-1}([3,3])$ must also equal $1$, by a similar calculation.

We still need $\zeta^{-1}(\underline{[1,2]})$ & $\zeta^{-1}([1,3])$

$x = 1$, $y = 2$:

$$\sum_{1 \le z \le 2} \zeta^{-1}([z,2]) = \delta([1,2])$$

i.e $\zeta^{-1}([1,2]) + \underbrace{\zeta^{-1}([2,2])}_{=1 \text{ by previous calculation}} = \delta([1,2]) = 0$

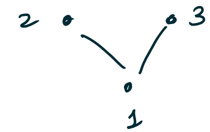
$\Rightarrow$ $\zeta^{-1}([1,2]) = 0 - \zeta^{-1}([2,2]) = -1$

So $\zeta^{-1}([1,2])$ must be $-1$.

Similarly, we can check that $\zeta^{-1}([1,3]) = -1$

⇝ Now we have produced a candidate $\zeta^{-1}$:

$\zeta^{-1}([a,a]) = 1$ for any $a \in P$



$\zeta^{-1}([a,b]) = -1$ for $a = 1$,
$b = 2$ or $3$.

Technically, we should check that
$(\underline{\zeta * \zeta^{-1}}) = (\zeta^{-1} * \zeta) = \delta$ on any interval $[x,y]$

But we've basically done this in the previous calculations

So for this poset, we have found an inverse!

** $\underline{\text{Thm}}$ : $\zeta$ is invertible for any finite poset.

The inverse $\zeta^{-1}$ has a special name: mu $\mu$

We'll come back to this theorem

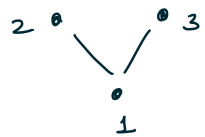
## * Matrix representation of $\mathcal{A}(P)$

Let $P$ be a finite poset.

We'll now write elements of $\mathcal{A}(P)$ as matrices.

Suppose $f \in \mathcal{A}(P)$; we'll construct associated $M_f$.

① Choose any labelling of $P$: $(p_1, p_2, \dots, p_n)$
(Any labelling is ok, but it's much better to choose a topological sorting)



② If $P$ has $n$ elements, make $n \times n$ matrix.

$$i \to \begin{array}{c} j \\ \downarrow \\ \left[\quad \bullet \quad\right] \end{array}$$

the $(i,j)^{th}$ entry of $M_f$
is $f([p_i, p_j])$ if $p_i \leq p_j$
is $0$ otherwise

## ** Example

$\delta \in \mathcal{A}(P)$

$$M_\delta = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

blue circled entries correspond to empty intervals

$\zeta \in \mathcal{A}(P)$ $\quad M_\zeta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mu = \zeta^{-1} \in \mathcal{A}(P)$

$$M_\mu = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from the calculation we did earlier.

** Thm: Addition and convolution of elements of $\mathcal{A}(P)$ correspond exactly to additions & matrix multiplications of the corresponding matrices.