MATH 2301

* Last time: Matrix representation of incidence algebra

Let P be a finite poset.
Choose a labelling (or ordering) of the elements of P
$(p_1, p_2, \cdots, p_n)$
Preferably, choose a topological sorting (matrix looks nicer this way: it is upper-triangular)

Given $f \in \mathcal{A}(P)$, we create a matrix M_f.

$$\begin{array}{c|c} & \begin{matrix} p_1 & \cdots & p_j & \cdots & p_n \end{matrix} \\ \begin{matrix} p_1 \\ \vdots \\ p_i \\ \vdots \\ p_n \end{matrix} & \begin{bmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & \bigcirc & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

$(i,j)^{th}$ entry is
- $f([p_i, p_j])$ if $[p_i, p_j] \neq \emptyset$
- 0 if $[p_i, p_j] = \emptyset$.

** Theorem: Let $f, g \in \mathcal{A}(P)$. Choose an ordering $(p_1, \cdots, p_n)$ of P. Let M_f, M_g be the associated matrices of f & g. Then:

(1) The matrix associated to $(f+g)$ is $M_f + M_g$:

$$M_{(f \underset{=}{+} g)} = M_f \underset{=}{+} M_g$$

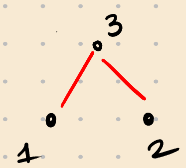
addition in $\mathcal{A}(P)$. matrix sum

(2) The matrix associated to $(f * g)$ is $M_f \cdot M_g$:

$$M_{(f * g)} = M_f \underset{=}{\cdot} M_g$$

convolution product matrix product

** Examples



$f = \zeta$ function

$g([x,y]) := x + y$.

$$M_f = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_g = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$(f * g)([x,y]) = \sum_{x \preceq z \preceq y} f([x,z]) \cdot g([z,y])$$

$$(M_f \cdot M_g)_{(i,j)} = \sum_{1 \le k \le n} \underbrace{(M_f)_{(i,k)}}_{\substack{= \\ f([p_i,\, p_k]) \\ \text{or } 0 \text{ if } [p_i, p_k] = \phi}} \cdot \underbrace{(M_g)_{(k,j)}}_{\substack{g([p_k, p_j]) \text{ or} \\ 0 \text{ if } [p_k, p_j] = \phi}}$$

$$(M_f \cdot M_g)_{(i,j)} = \sum_{\substack{p_i \preceq p_k \\ p_k \preceq p_j}} f([\underbrace{p_i}_{x}, \underbrace{p_k}_{z}]) \cdot g([\underbrace{p_k}_{z}, \underbrace{p_j}_{y}])$$

Compare w/ previous:

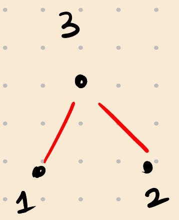
$$(f * g)([x,y]) = \sum_{x \preceq z \preceq y} f([x,z]) \cdot g([z,y])$$

This comparison proves part (2) of the theorem

$$M_f = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_g = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$(M_f \cdot M_g) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 2 & 0 & 10 \\ 2 & 0 & 4 & 11 \\ 3 & 0 & 0 & 6 \end{array}$$

it simultaneously gives all values of $(f*g)$ on all intervals



$$\Rightarrow (f*g)([1,3]) = 10$$
$$(f*g)([3,3]) = 6$$

Theorem part (1) is easier:

$$(f+g)([x,y]) = f([x,y]) + g([x,y])$$

$$(M_f + M_g)_{(i,j)} = (M_f)_{(i,j)} + (M_g)_{(i,j)}$$
$$= f([p_i, p_j]) + g([p_i, p_j])$$
[or zero if $[p_i, p_j] = \emptyset$]

$$\Rightarrow M_{f+g} = M_f + M_g.$$

** The function μ [Möbius function]

Recall the ζ function in $\mathcal{A}(P)$.

$$(M_\zeta)_{(i,j)} = \begin{cases} 1 & \text{if } [p_i, p_j] \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Suppose we have a topological sorting of P
$(p_1, \cdots, p_n)$
if $p_i \leq p_j$ then $i \leq j$.

Let $f \in \mathcal{A}(P)$

Suppose that $i > j$. This means that $p_i \not\preceq p_j$ [either they are incomparable, or $p_j \prec p_i$]

In any case, $[p_i, p_j] = \phi$.

$(M_f)_{(i,j)} = 0$.

Given a topological sort, we conclude that all entries below the diagonal in M_f are zero:

$$\left.\begin{bmatrix} & & ? & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \text{upper-triangular matrix.}$$

Back to ζ: given a topological sort,

$$M_\zeta = \begin{bmatrix} 1 & & & ? \\ 0 & 1 & & \\ 0 & 0 & \ddots & \\ 0 & 0 & \cdots & \\ 0 & 0 & \cdots\; 0 & 1 \end{bmatrix}$$

1s & 0s depending on which intervals are $\neq \phi$.

1s on the diagonal

0s below the diagonal

From this structure of M_ζ, it is easy to tell whether M_ζ is invertible.

** Theorem: An upper-triangular matrix is invertible if and only if all of its diagonal entries are non-zero.

$\Rightarrow$ (Looking at M_ζ): M_ζ is invertible.

$\Rightarrow$ ζ is always invertible for any poset!

(But how to find it? ...)

** **Def** : The Möbius function $\mu \in \mathcal{A}(P)$ is the inverse of ζ :

$$\mu * \zeta = \delta = \zeta * \mu$$

i.e. $M_\mu \cdot M_\zeta = I = M_\zeta \cdot M_\mu$

$$I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix}$$

** **Theorem** : An element $f \in \mathcal{A}(P)$ is invertible if and only if all diagonal entries of M_f are non-zero.

That is, if and only if $f([x,x]) \neq 0$ for every $x \in P$.