MATH 2301

* Recap : Matrix representation

Let $(P, \preceq)$ be a poset. Fix an ordering $(p_1, \cdots, p_n)$ of P.

Given $f \in \mathcal{A}(P)$, we have M_f, an $n \times n$ matrix.

** Key points:

- $(M_f)_{(i,j)} = \begin{cases} f([p_i; p_j]) & \text{if } [p_i; p_j] \neq \phi \\ 0 & \text{otherwise} \end{cases}$

- If $(p_1, \cdots, p_n)$ is a topological sort, then M_f is upper-triangular. $\begin{bmatrix} & * \\ 0 & \end{bmatrix}$

- $M_{f+g} = M_f + M_g$

  $M_{f*g} = M_f \cdot M_g$

- $M_{(f^{-1})} = (M_f)^{-1}$ ← if f is invertible as an element of $\mathcal{A}(P)$.

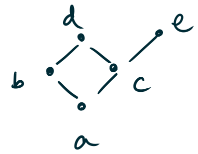
- Theorem : f is invertible if and only if all diagonal entries of M_f are non-zero. Equivalently, if and only if $f([x,x]) \neq 0$ for every $x \in P$.

** The Möbius function μ:

Let $(P, \preceq)$ be a finite poset.

Def: The Möbius function μ is the inverse of $\zeta \in \mathcal{A}(P)$.

*** Computing μ.



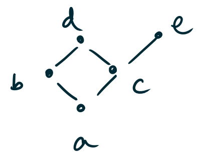
$\mu * \zeta = \delta$.

$(\mu * \zeta)([x,y]) = \delta([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else.} \end{cases}$

(1) Case 1: $\mu([x,x]) = ?$ (x some element of P)

$(\mu * \zeta)([x,x]) \underset{\substack{\uparrow \\ \text{from formula}}}{=} \mu([x,x]) \cdot \underbrace{\zeta([x,x])}_{=1} = \delta([x,x]) = 1$

$\Rightarrow \mu([x,x]) = 1$



(2) Case 2: $\mu([x,y])$ for $y \neq x$.

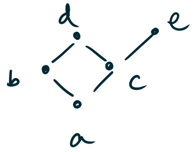
E.g. $\mu([a,c])$

$(\mu * \zeta)([a,c]) = \sum_{a \preceq z \preceq c} \mu([a,z]) \underbrace{\zeta([z,c])}_{=1} = \delta([a,c]) = 0$

$= \mu([a,a]) + \mu([a,c]) = 0. \Rightarrow \mu([a,c]) = -1.$

Similarly, we have an equation (from $\mu([a,e])$):

$\underbrace{\mu([a,a])}_{=1} + \underbrace{\mu([a,c])}_{=-1} + \mu([a,e]) = 0 \Rightarrow \mu([a,e]) = 0.$



$\mu([a,d])$?

Equation:

$$\underbrace{\mu([a,a])}_{=1} + \underbrace{\mu([a,b])}_{=-1} + \underbrace{\mu([a,c])}_{=-1} + \mu([a,d]) = 0.$$

$$\Rightarrow \mu([a,d]) = 2 - 1 = 1.$$

In general: suppose $x \neq y$

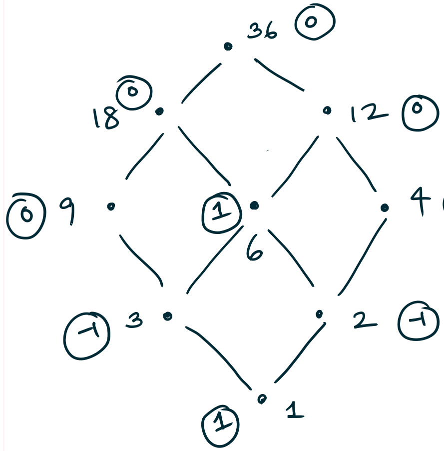
$$(\mu * \zeta)([x,y]) = \delta([x,y]) = 0.$$

$$\sum_{x \leq z \leq y} \mu([x,z]) \underbrace{\zeta([z,y])}_{=1} = 0.$$

$$\Rightarrow \mu([x,y]) + \sum_{x \leq z \lneq y} \mu([x,z]) = 0.$$

$$\Rightarrow \mu([x,y]) = - \sum_{x \leq z \lneq y} \mu([x,z])$$ recursive formula.

** Example: Divisor poset



Let us compute $\mu([1,-])$

Blue values represent $\mu([1,x])$ at each x.

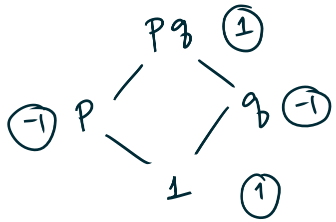
* There is in fact an intrinsic formula for $\mu([a,b])$ in this case.

** Observe [return to this later]

- If p is a prime, then $\mu([1,p]) = -1$.

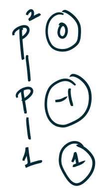
} portion of divisor poset between 1 & p

- If p, q are different primes, then:



$\mu([1,pq]) = 1.$

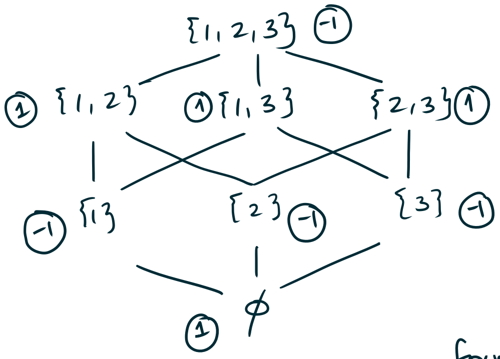
- If p is a prime, then:



$\mu([1,p^2]) = 0$

** Example : Subset poset

$S = \{1,2,3\}$, $P(S) = \{A \subseteq S\}$



Again, let's compute

$\mu([\phi, -])$

Blue values indicate this

* Again, there is an intrinsic formula for $\mu([A,B])$ in this case.

* Proposition : In a subset poset,

(1) $\mu([\phi, A]) = (-1)^{|A|}$ ← number of elements in A

(2) $\mu([A,B]) = (-1)^{|B \setminus A|}$ ← number of elements in the set difference.

** Products of posets

Def : Let $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ be posets.

The product poset is the set $P_1 \times P_2$, with the order relation being

$(a,b) \preceq (c,d)$ if $a \preceq_1 c$ and $b \preceq_2 d$

** Examples

(1) P_1 :

P_2 :

$P_1 \times P_2$: