MATH 2301

* Exam this evening @ 6:30 pm. (unless otherwise arranged)
  - Zoom details on Wattle
  - Keep video ON and mic MUTED.
  - Communicate with invigilators via zoom chat only.

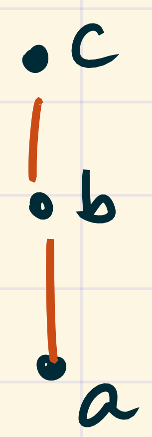
* Continued: products of posets

** Def: Let $(P_1, \preceq_1)$ & $(P_2, \preceq_2)$ be posets. The product poset is defined as the set $P_1 \times P_2$ with the relation

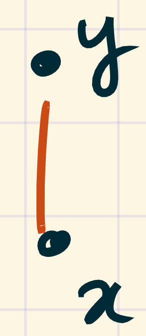
$(a,b) \preceq (c,d)$ if $a \preceq_1 c$ and $b \preceq_2 d$.

** Example

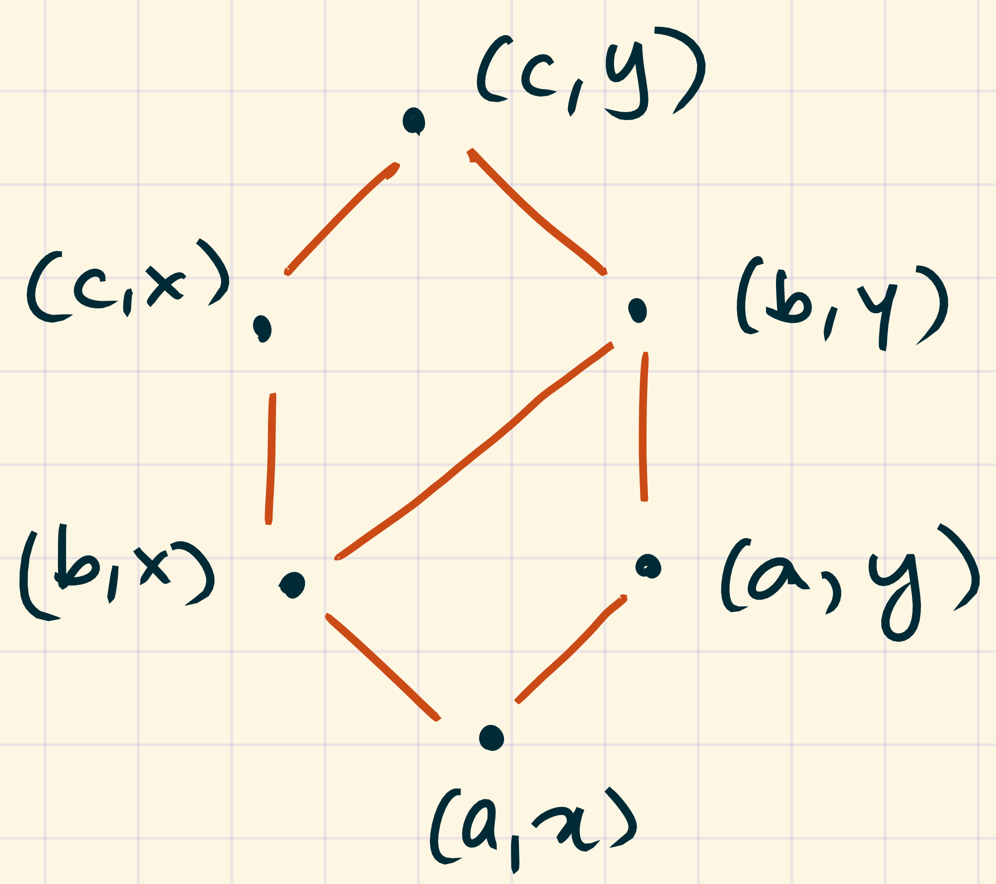
(1) $P_1 =$



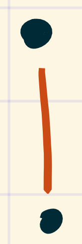
$P_2 =$



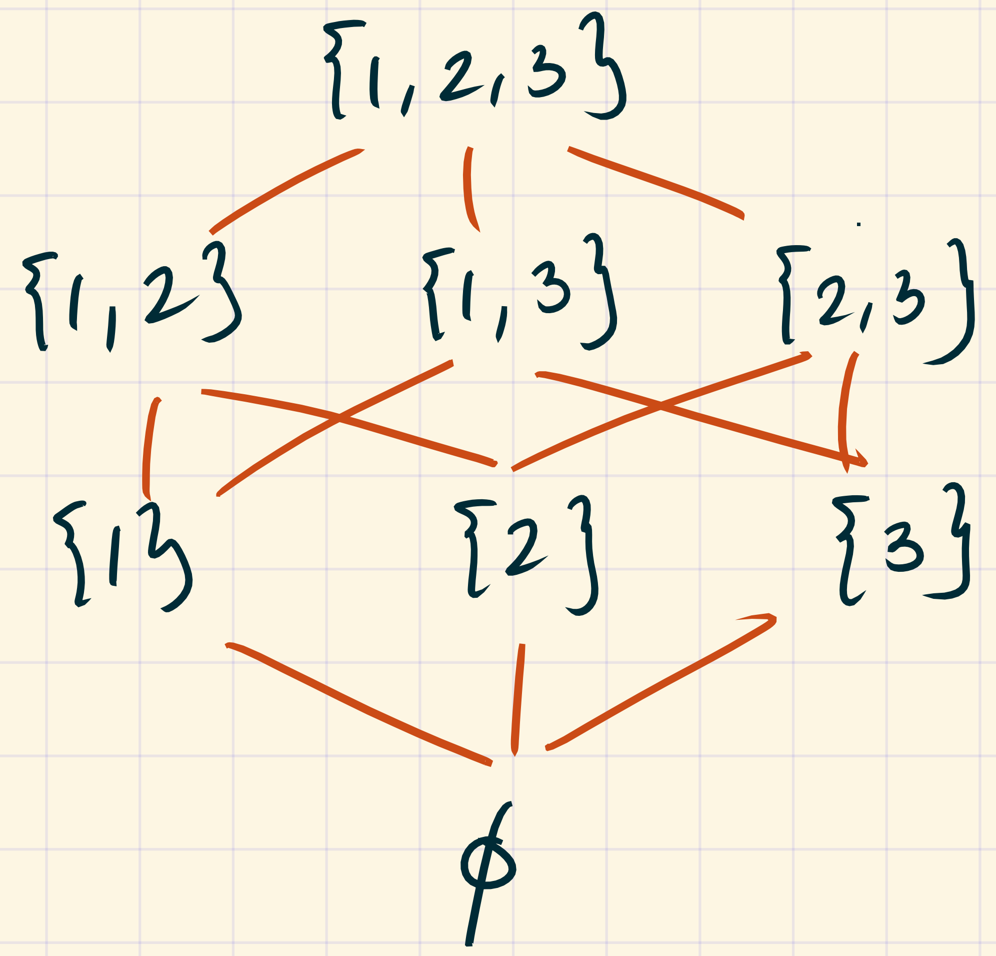
$P_1 \times P_2 =$



(2) The subset poset

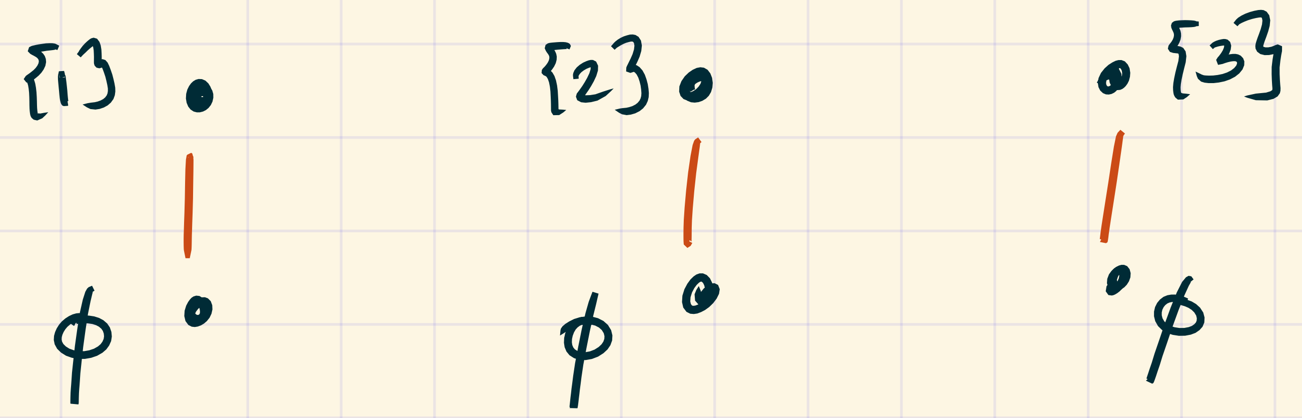
Let $P =$ 

$A = \{1,2,3\}$

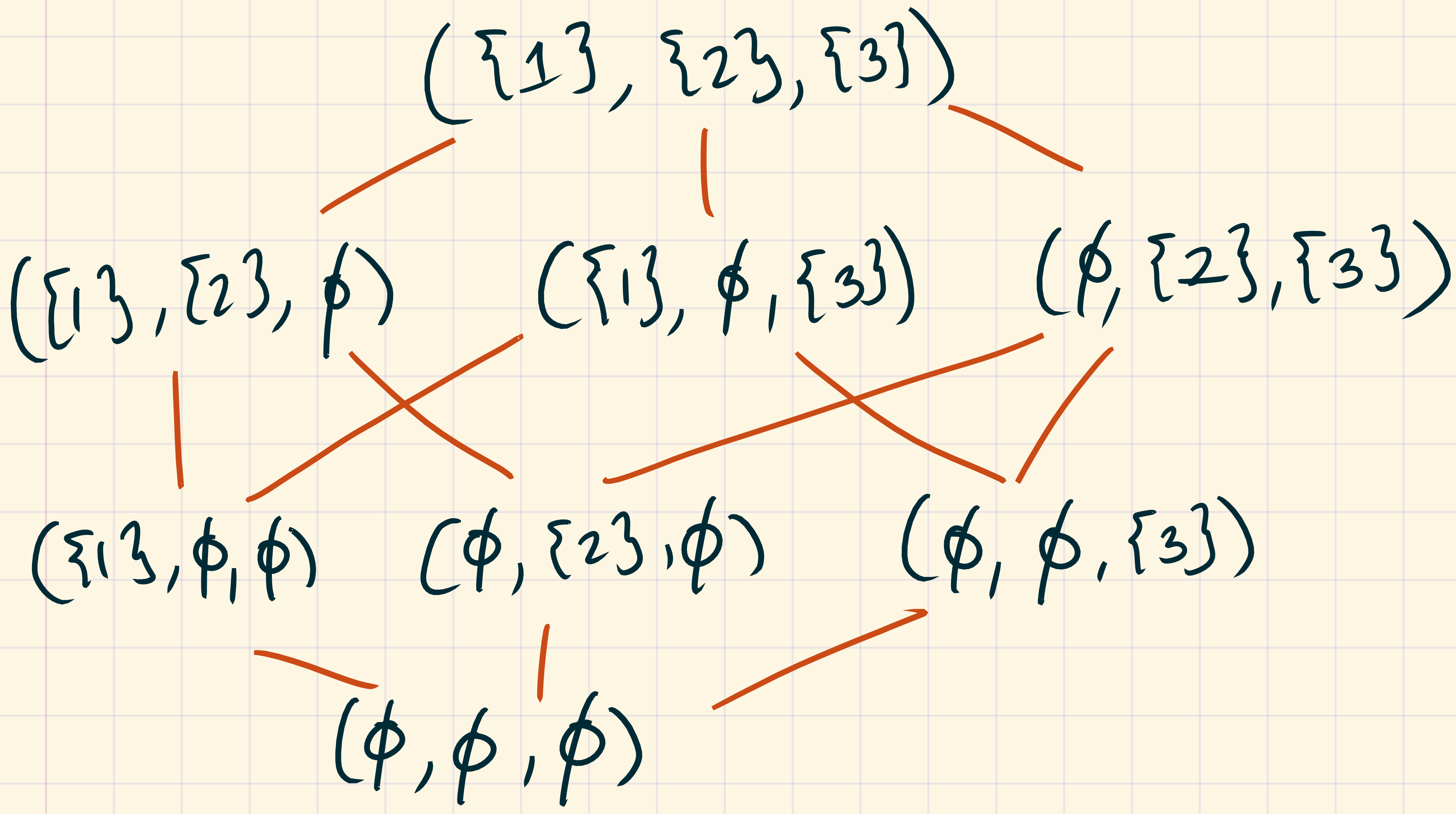


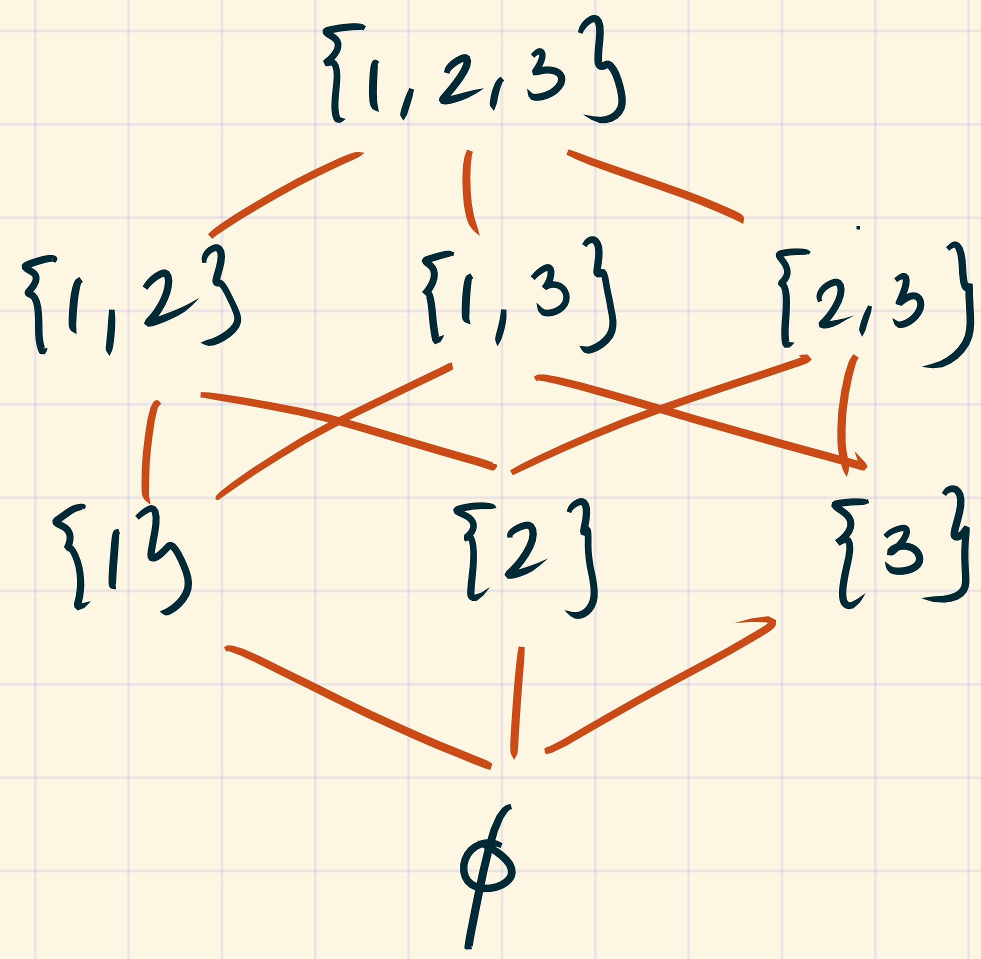
Claim: The subset poset of A has the same Hasse diagram as $(P \times P \times P)$

Label the 3 copies of P as follows:



Let's draw $P \times P \times P$. Compare w/ previous.

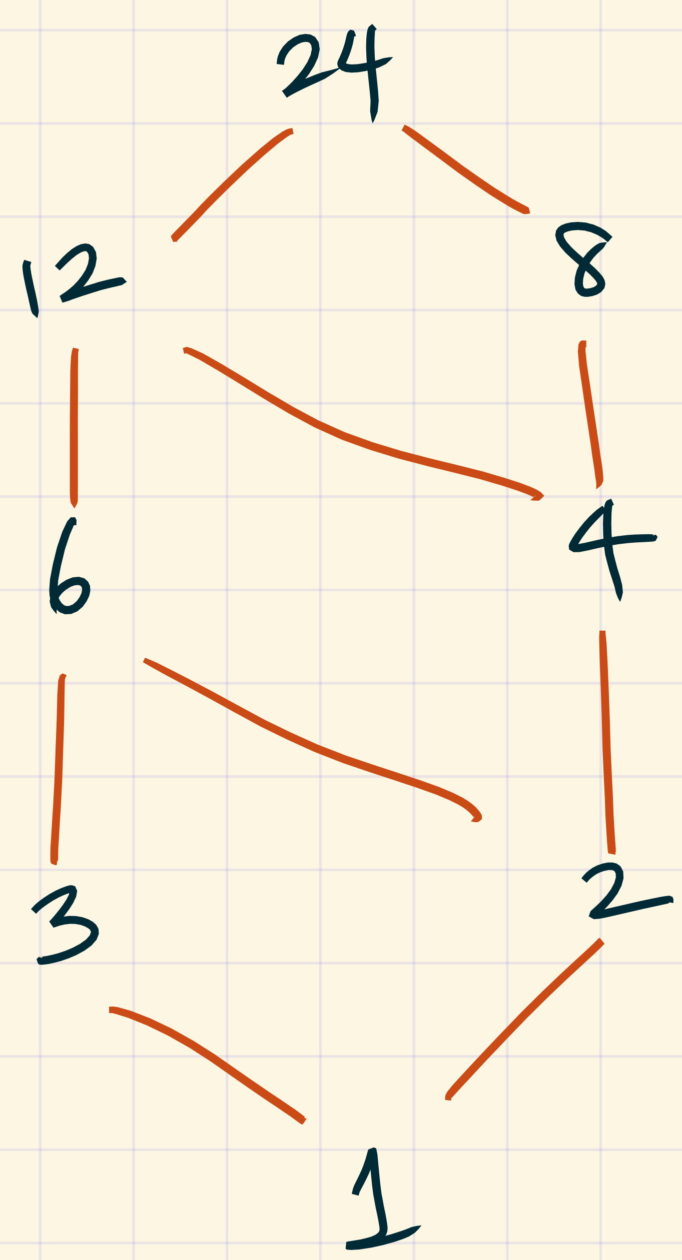


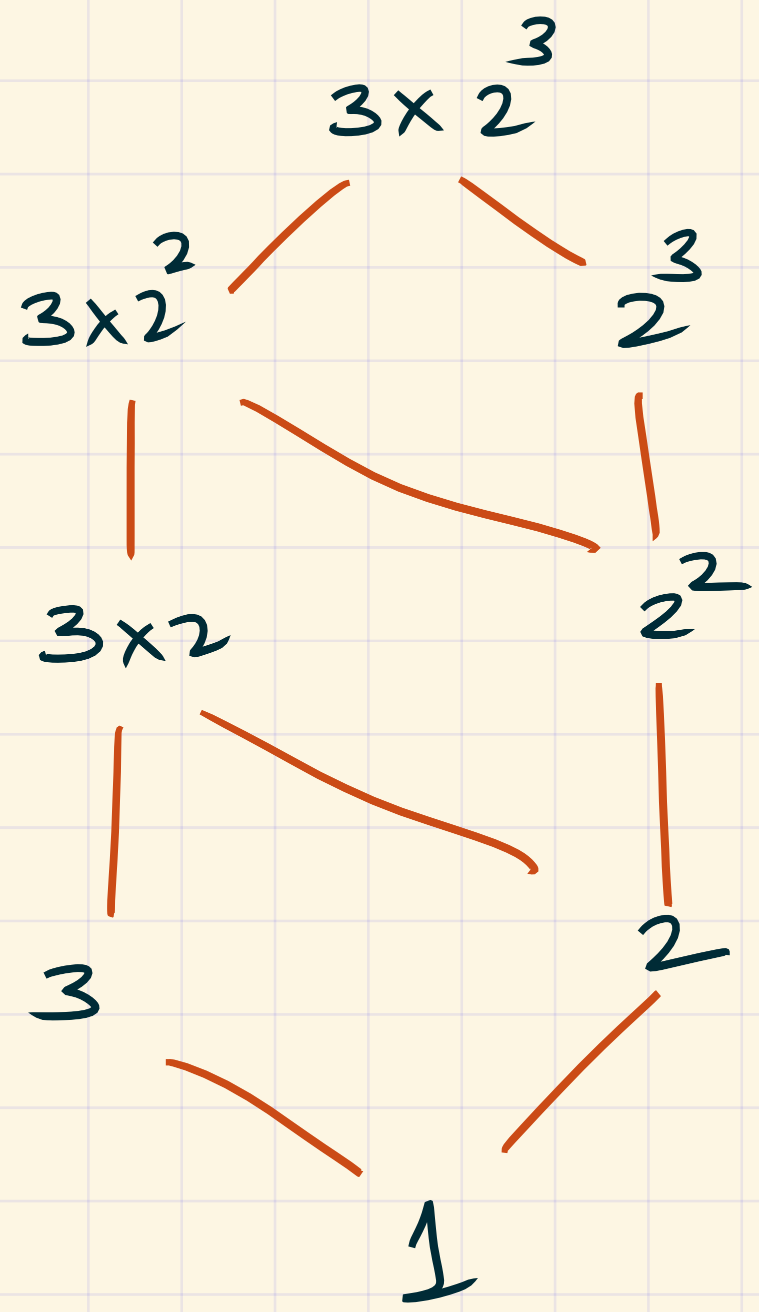


Left hand side $\longrightarrow$ right hand side is by taking set union of all coordinates.

** **Theorem**: Let A be a set of n elements. Then the subset poset of A has the same Hasse diagram (same shape) as the product $(\underbrace{P \times P \times \cdots \times P}_{n \text{ times}})$

(3) The divisor poset

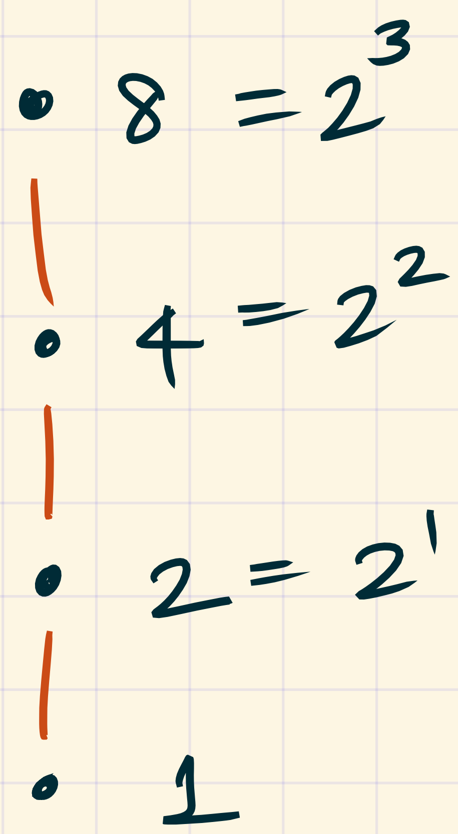




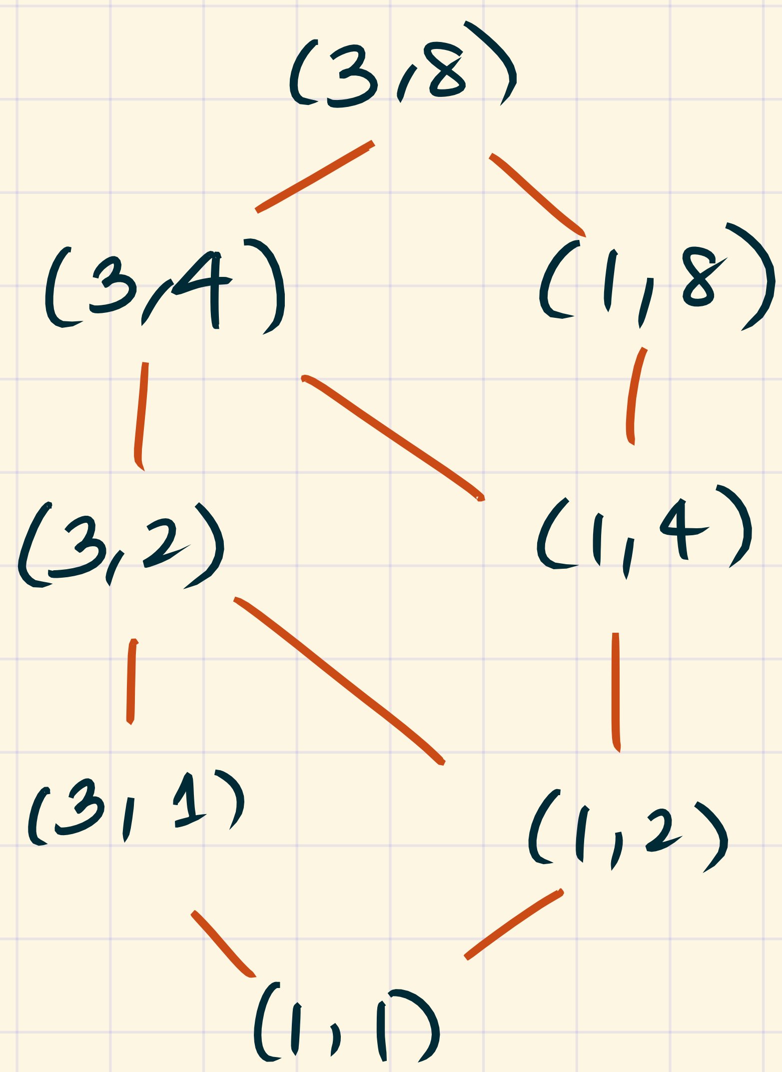
prime power decomposition

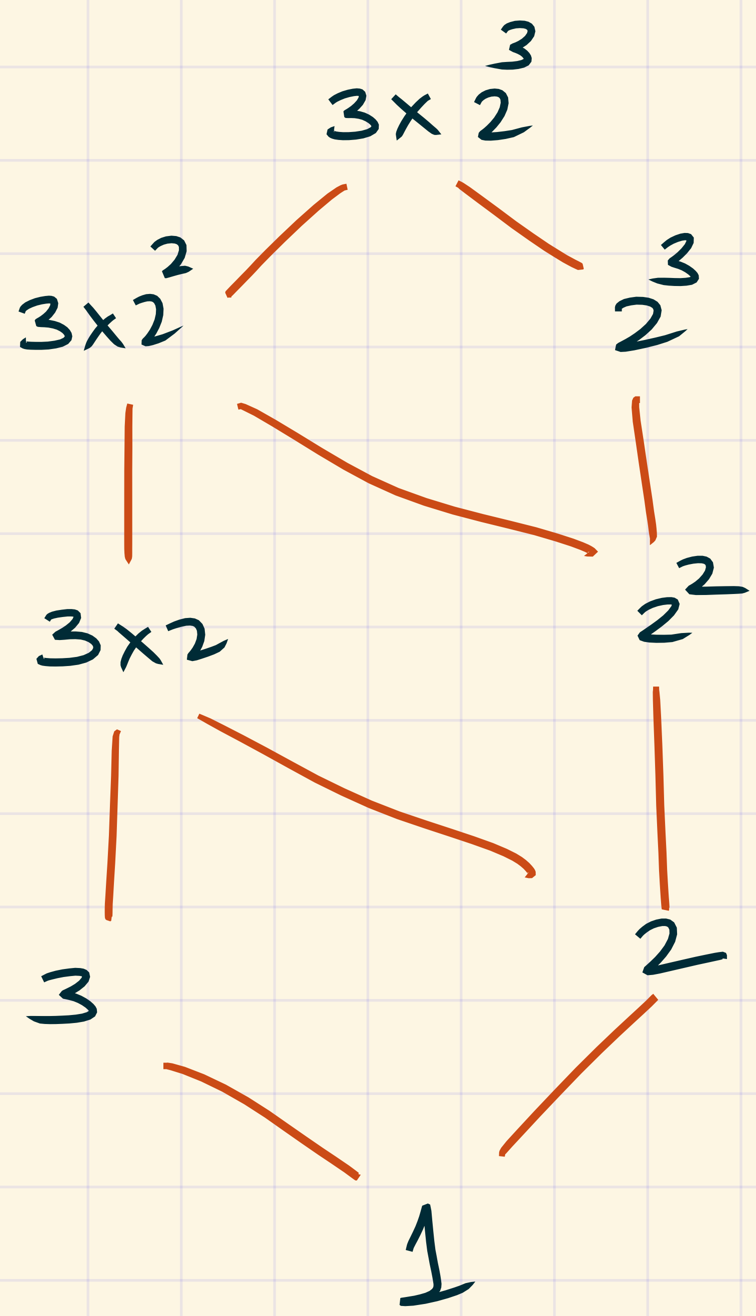
$P_1 =$

$P_2 =$



$P_1 \times P_2$:





LHS → RHS by taking product of the coordinates.

** <u>Theorem</u> : Let $m \geq 1$ be an integer. Suppose

$$m = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$$ be its prime power decomposition: $p_1, \ldots, p_k$ all prime, and $p_1 < p_2 < \cdots < p_k$

Let $P_1, P_2, \ldots, P_k$ be the divisor posets of $p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}$.

Then the shape of the Hasse diagram for the divisor poset of m is the same as that for $P_1 \times P_2 \times \cdots \times P_k$.

** <u>Observe</u> : The divisor poset of any prime power p^n is a linear (total order):

** Let's go back to computing μ.
(We'll use the previous observation)

Recall : μ is the inverse of ζ.

** <u>Theorem</u>: Let $(P_1, \preceq)$ and $(P_2, \preceq)$ be posets. Let μ_1 and μ_2 be the μ functions of P_1 & P_2 respectively.
Let μ be the μ-function for $P_1 \times P_2$. Then, for any interval $[(a,b),(c,d)]$ in $P_1 \times P_2$, we have:

$$\mu([(a,b),(c,d)]) = \mu([a,c]) \cdot \mu_2([b,d])$$

** <u>Corollary</u>

(1) Let A be a set w/ n elements. Let $X \subseteq Y$ be subsets of A.

$$\boxed{\mu([X,Y]) = (-1)^{|Y \setminus X|}}$$

<u>E.g.</u> $X = \{3\}$ & $Y = \{1,2,3\}$

$$\updownarrow \qquad\qquad \updownarrow$$

$$(\phi, \phi, \{3\}) \qquad (\{1\}, \{2\}, \{3\})$$

$$\mu([X,Y]) = \underbrace{\mu([\phi, \{1\}])}_{=-1} \cdot \underbrace{\mu([\phi, \{2\}])}_{=-1} \cdot \underbrace{\mu([\{3\},\{3\}])}_{=1}$$

$$= 1$$

(2) Let m, n be positive integers.

$$n = p_1^{a_1} \cdots p_k^{a_k}, \quad m = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}.$$

$$\mu([1, n]) = \mu([1, p_1^{a_1}]) \cdots \mu([1, p_k^{a_k}])$$

$$\mu([n, m]) = \mu([p_1^{a_1}, p_1^{b_1}]) \cdots \mu([p_k^{a_k}, p_k^{b_k}])$$

(Let's finish this on Wed.)