MATH 2301

* Assignment 5 will be posted by tomorrow & due next Friday.
* Mid-semester class survey now open

* Möbius functions on product posets

Let μ_1, μ_2 be the Möbius functions for P_1 & P_2.

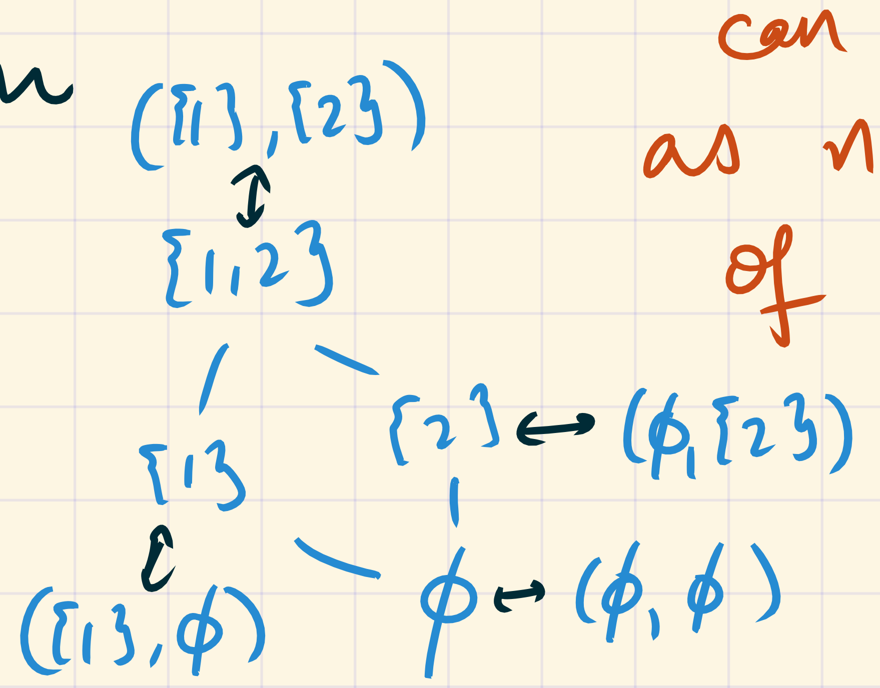
Let μ = Möbius functions for $P_1 \times P_2$

** **Theorem**: Let $a, c \in P_1$ with $a \preceq_1 c$. Let $b, d \in P_2$ with $b \preceq_2 d$. Then:

$$\mu([(a,b),(c,d)]) = \mu_1([a,c]) \cdot \mu_2([b,d]).$$

** **Consequences**

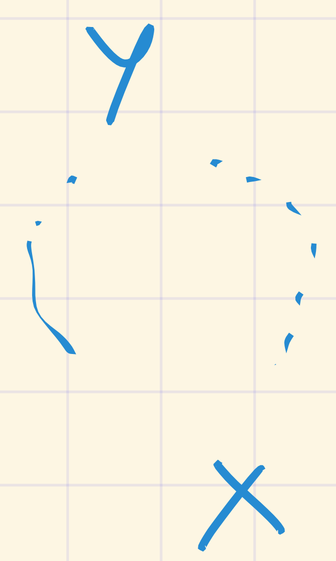
(1) If A is a set with n elements, the $\underbrace{\text{subset poset}}_{\text{can be written as } n\text{-fold product of}}$ of A has the μ function



$$\mu([\phi, X]) = (-1)^{|X|}$$

set minus = set difference

$$\mu([X,Y]) = (-1)^{|Y \setminus X|}$$



* Every element of Y that's not in X contributes a (-1) to the product

(2) Let $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

$n = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$

$\Big\}$ $p_1 < p_2 < \cdots < p_k$ are distinct primes

$a_i, b_i \geq 0$ are integers.

such that $m \mid n$.

Example : $m = 6 = 2^1 \cdot 3^1 = 2^1 \cdot 3^1 \cdot 5^0$

$n = 60 = 2^2 \cdot 3^1 \cdot 5^1$

Observe that $m \mid n$ means that $a_i \leq b_i$ for each i

$$\mu([m,n]) = \mu([p_1^{a_1}, p_1^{b_1}]) \cdot \mu([p_2^{a_2}, p_2^{b_2}]) \cdots$$

$$\cdots \mu([p_k^{a_k}, p_k^{b_k}])$$

E.g.

$$\mu([6,60]) = \mu([2^1, 2^2]) \cdot \mu([3^1, 3^1]) \cdot \mu([5^0, 5^1])$$

$\mu([2, 2^2]) = (-1)$ $\Big\{$ 2^2 | 2

3^1 $\big]$ $\mu([3,3]) = 1$

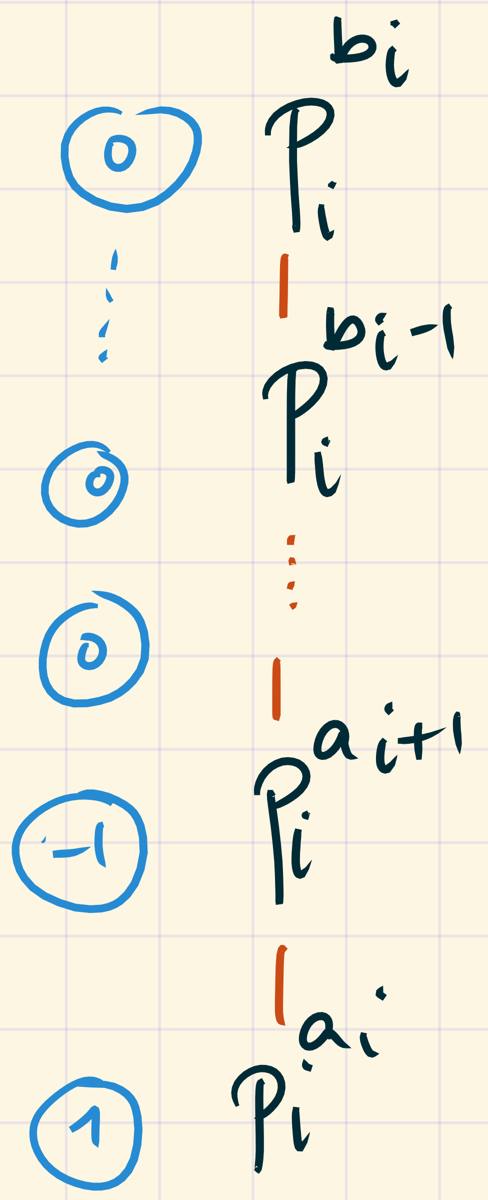
5^1 | $5^0 = 1$ $\quad \mu([1,5]) = (-1)$

$\Rightarrow \mu([6,60]) = 1$.

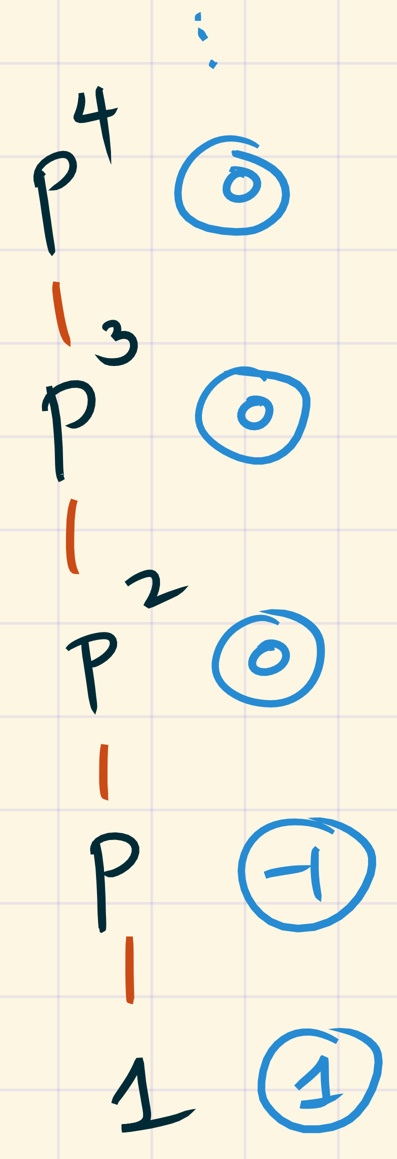
In the general case, for every i, we'll have to compute $\mu([p_i^{a_i}, p_i^{b_i}])$

$\mu([p_i^{a_i}, -])$

(record the values in blue)



E.g.



$\mu([1, p^x])$ for various x

(record in blue)

$$\Rightarrow \mu([p_i^{a_i}, p_i^{b_i}]) = \begin{cases} 1 & \text{if } b_i = a_i \\ (-1) & \text{if } b_i = a_i + 1 \\ 0 & \text{if } b_i > a_i + 1 \end{cases}$$

$$\mu([m,n]) = \mu([p_1^{a_1}, p_1^{b_1}]) \cdot \mu([p_2^{a_2}, p_2^{b_2}]) \cdots$$

$$\cdots \mu([p_k^{a_k}, p_k^{b_k}])$$

$\Rightarrow$ This tells us a concise formula for $\mu([m,n])$

It depends on the divisors of (n/m).

E.g. $\left.\begin{array}{l} m = 6 \\ n = 60 \end{array}\right\} \; n/m = 10 = 2 \times 5$

Theorem: Let $m \mid n$. The value of $\mu([m,n])$ is:

$$\begin{cases} 0 & \text{if } n/m \text{ is divisible by the square of a prime} \\ (-1)^e, & \text{where } e = \#\text{ distinct prime factors of } n/m. \end{cases}$$

Examples

(1) $\mu([6,60]) = (-1)^2 = 1$

$60/6 = 10 = 2 \times 5$

$\Rightarrow$ 2 distinct prime factors

(2) $\mu([1,60]) = 0$

$4 \mid 60$, 4 is the square of a prime

(3) $\mu([2,60]) = (-1)^3 = (-1)$

$60/2 = 30 = 2 \times 3 \times 5$

$\Rightarrow$ 3 distinct prime factors.

(4) $\mu([2,36]) = 0$

$36/2 = 18$ is divisible by 9, which is the square of a prime.

* <u>One-sided convolution and matrices.</u>

Fix a poset P and an ordering $(a_1, \cdots a_n)$.

Given $f, g \in \mathcal{A}(P)$, we have $n \times n$ matrices M_f, M_g, such that

(1) $M_{f+g} = M_f + M_g$

(2) $M_{(f*g)} = M_f \cdot M_g$.

Suppose $p : P \to \mathbb{R}$.

We can construct a vector $v_p = \begin{bmatrix} p(a_1) \\ p(a_2) \\ \vdots \\ p(a_n) \end{bmatrix}$ ← $n \times 1$ vector

We can also construct $v_p^t = [p(a_1)\ p(a_2) \cdots p(a_n)]$ ← $1 \times n$

** <u>Theorem</u>: If $f \in \mathcal{A}(P)$, and $p : P \to \mathbb{R}$, then:

(1) $v_{(f*p)} = \underbrace{M_f}_{n \times n} \cdot \underbrace{v_p}_{n \times 1}$

(2) $v_{(p*f)}^t = \underbrace{v_p^t}_{1 \times n} \cdot \underbrace{M_f}_{n \times n}$

One-sided convolution can be computed using matrix products.

* The inclusion-exclusion principle

Q: How many positive integers from 1 to 100 are not divisible by 2, 3, or 5 ?

A: Step 0 : Remove all multiples of 2 ← 50
" " " " 3 ← 33
" " " " 5 ← 20

Get $100 - 50 - 33 - 20$
$= 100 - (103) = -3$!!
[Because we double counted ...]

** Inclusion-exclusion in terms of poset functions