MATH 2301

* Last time: inclusion-exclusion principle

Q: How many integers from 1 to 100 are not divisible by 2, 3, or 5?

(a) divisible by 2 — 50

(b) divisible by 3 — 33

(c) divisible by 5 — 20

** First approximation: $100 - (50+33+20) = -3$ not the final answer of course.

This is wrong because:

- multiples of 2 & 3 [aka multiples of 6] were subtracted twice.
- multiples of 15 } also subtracted twice.
- multiples of 10 }

** Second approximation to answer: add in

- multiples of 6 — 16 = integer part of $\frac{100}{6}$
- multiples of 15 — 6
- multiples of 10 — 10

$\leadsto (-3) + (16+6+10) = 29.$

some of these were double-counted: multiples of 30

** Final answer: $29 - [\text{multiples of } 30] = 29 - 3 = 26.$

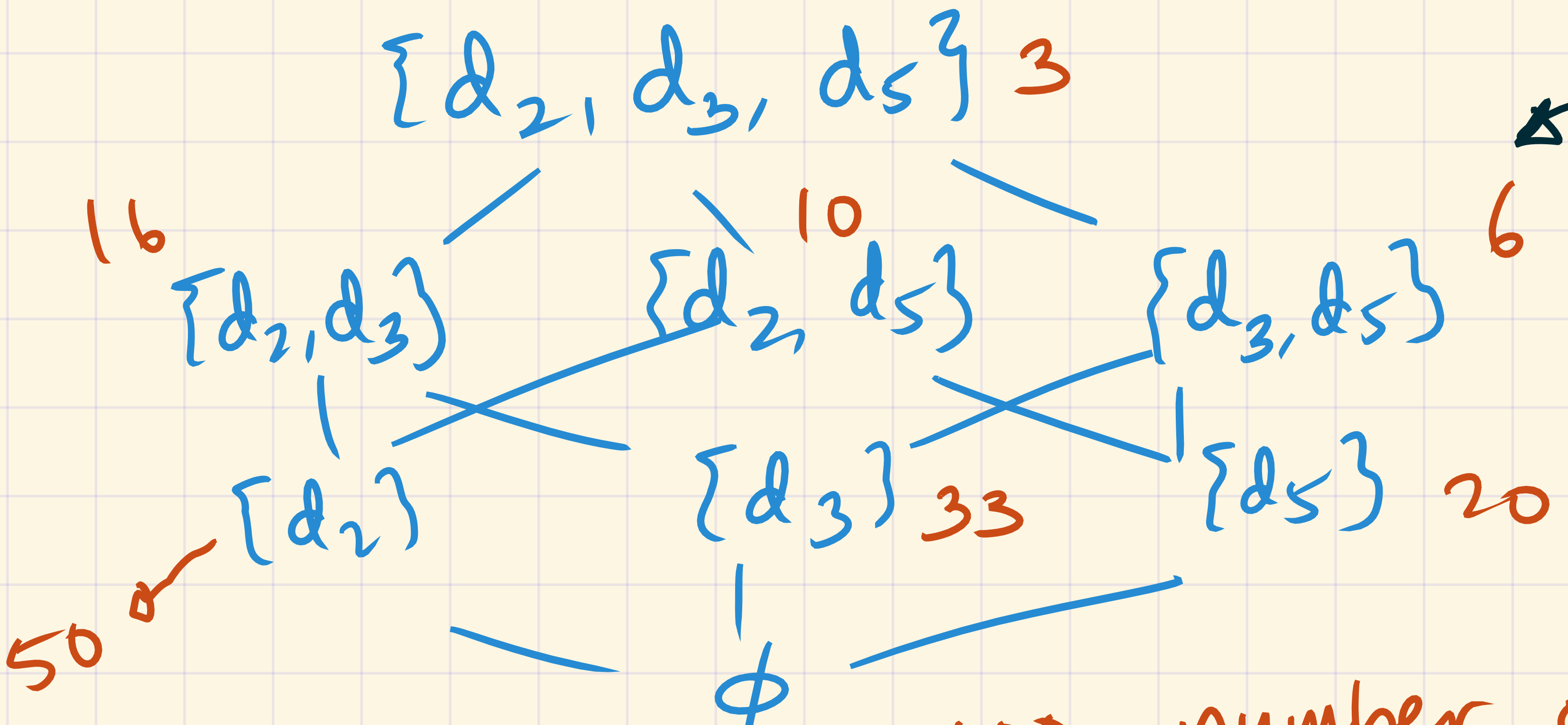
** Principle of inclusion-exclusion (PIE) on the subset poset

Let $s_1, s_2, \cdots, s_k$ be statements or properties that are or are not true about elements in a set A.

divisibility by 2, 3, 5 respectively. $\{1, 2, \ldots, 100\}$

Consider P = subset poset of $\{s_1, \cdots, s_k\}$, and two functions on P:

(1) $p : P \to \mathbb{R}$ is defined as follows. If $X \in P$, $p(X)$ = number of elements of A that satisfy $\boxed{\text{at least}}$ all the properties in X.

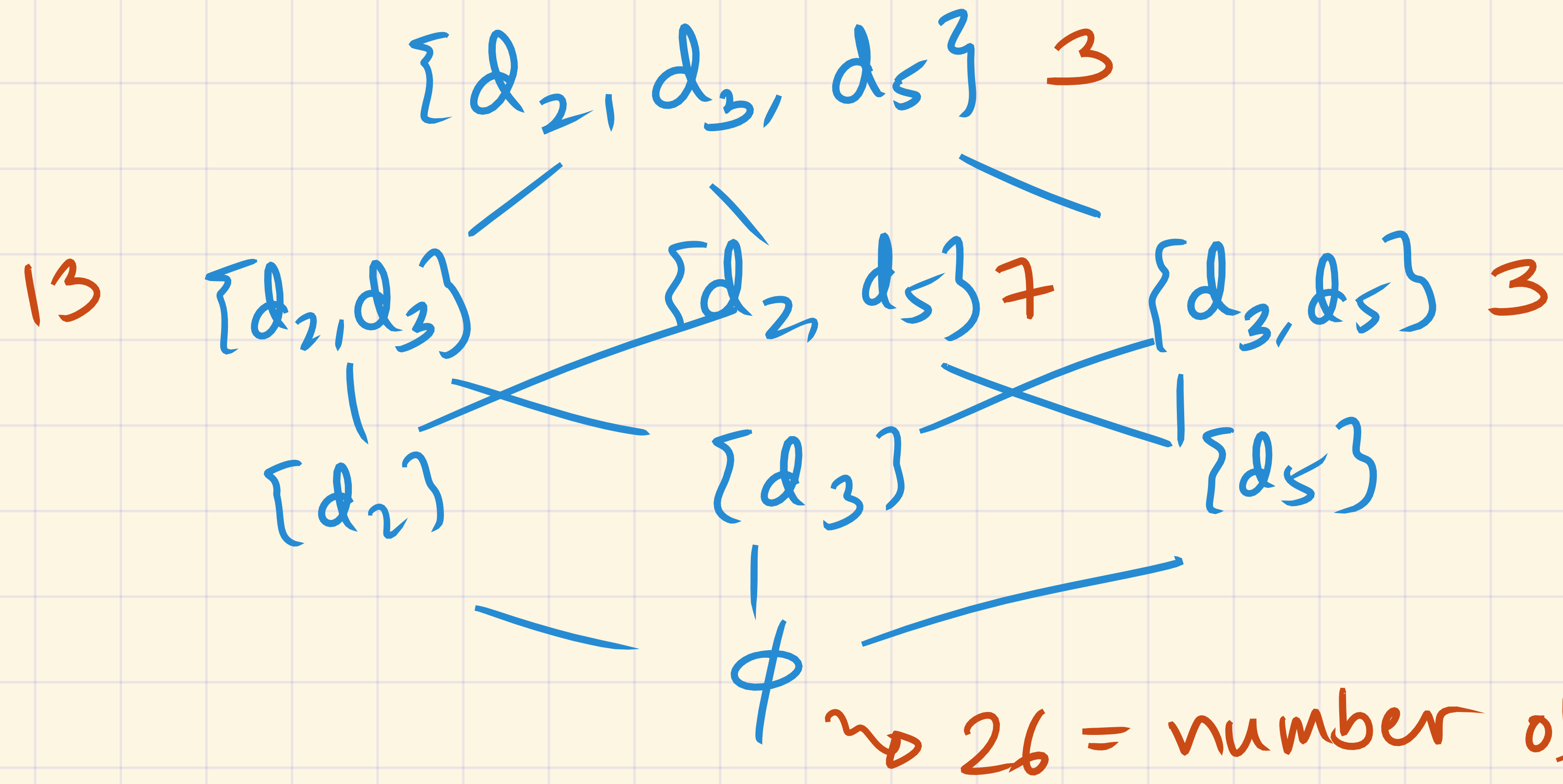


$\Big\{$ $p(x)$ for previous example.

$\phi \to 100$ = number of elts of A that satisfy at least none of the properties.

(2) $q : P \to \mathbb{R}$. If $X \in P$, then

$q(X)$ = number of elements in A that satisfy $\boxed{\text{exactly}}$ the properties in X, and do not satisfy the properties not in X.



Values of $q(X)$??

$\rightsquigarrow$ 26 = number of elements of A that satisfy none of the properties

Note that if $X \in P$, then:

$$p(X) = \sum_{X \preceq Y} q(Y)$$

(i.e. $X \subseteq Y$)

[In order to count in $p(X)$, you must satisfy everything in X, and possibly other properties $\rightsquigarrow$ this is captured by adding $q(Y)$ for every $Y \geq X$]

$$p(X) = \sum_{X \preceq Y} q(Y) = \sum_{X \preceq Y} \zeta([X,Y]) \cdot q(Y)$$

$$\Rightarrow \quad p(X) = (\zeta * q)(X) \quad , \text{ i.e. } \boxed{p = (\zeta * q)}$$

** Notice : ζ is invertible ; $\zeta^{-1} = \mu$.

Using $p = \zeta * q$, we get

$$\mu * p = \mu * (\zeta * q) = (\mu * \zeta) * q.$$

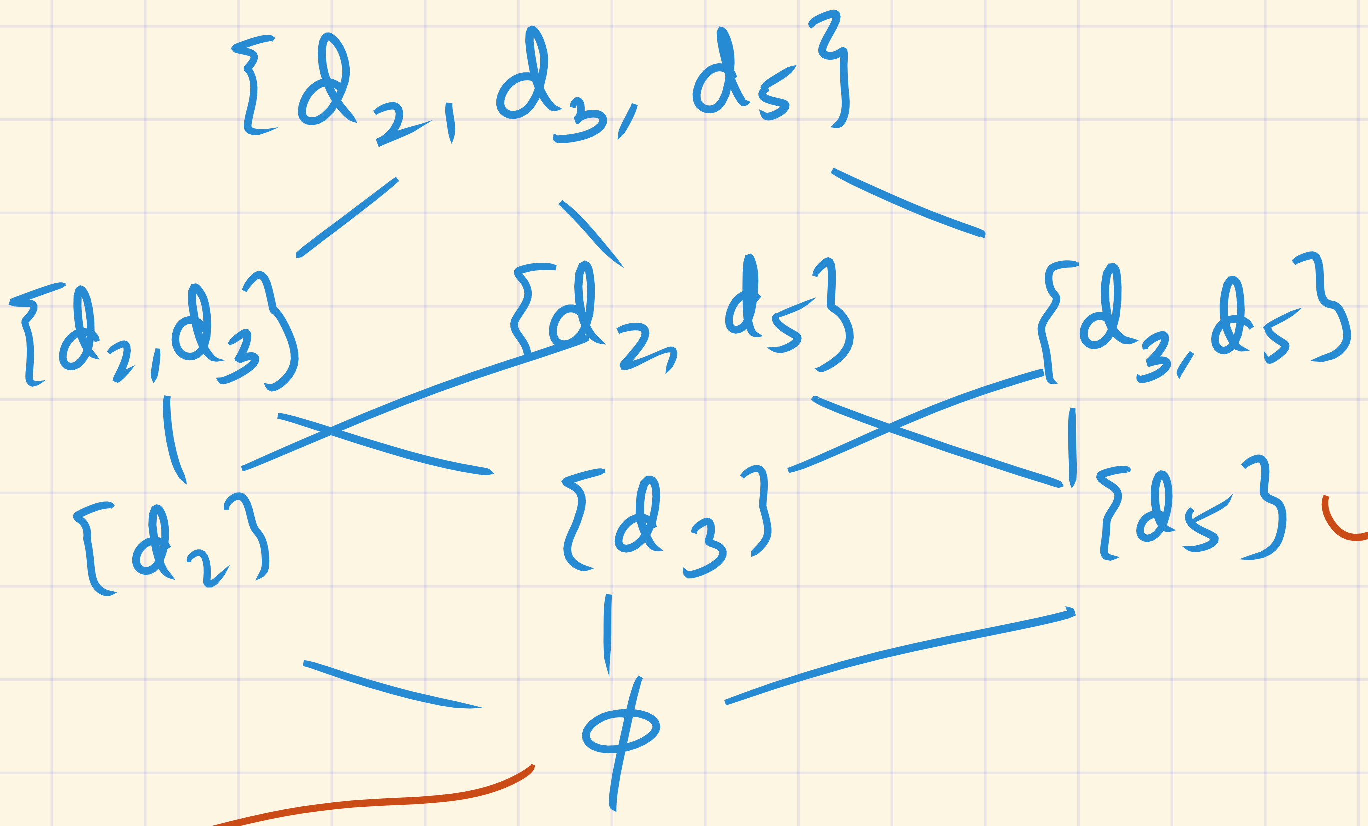
$$\mu * p = \delta * q = q.$$

$$\boxed{q = \mu * p}$$

So, if $X \in P$:

$$q(X) = (\mu * p)(X) = \sum_{X \preceq Y} \mu([X,Y]) \cdot p(Y)$$

$$q(X) = \sum_{X \preceq Y} (-1)^{|Y \setminus X|} \, p(Y)$$



$\{d_5\} \to 20 - 10 - 6 + 3 = 7$

$\mu([\phi, \{d_3, d_5\}]) \cdot p(\{d_3, d_5\})$

$$\underbrace{100}_{= p(\phi)} - 50 - \underbrace{33}_{p(\{d_3\})} - 20 + 16 + 6 + 10 - 30$$

$\mu([\phi, \{d_3\}]) = -1$

(PIE, restated as Möbius inversion)

** PIE (and Möbius inversion) on general posets.

Let P be a poset, and $p, q, r : P \to \mathbb{R}$ such that

$$p(x) = \sum_{x \preceq z} q(z) = (\zeta * q)(x)$$

$$p(x) = \sum_{z \preceq x} r(z) = (r * \zeta)(x)$$

the formulas can be "inverted" by multiplying by μ on left/right respectively.