MATH 2301

* Nim ordering clarification

You can consider the heaps to be un-ordered.

But keep multiplicity!

E.g. $(1,1,2) = (2,1,1) = (1,2,1)$ but $(1,1,2) \neq (1,2)$.

* Nim strategy

** Experiments with XOR.

XOR = exclusive OR, is an operation on binary strings, defined as follows.

Examples

$$\begin{array}{r} 1011_2 \\ \oplus\ 101_2 \\ \hline 1110_2 \end{array}$$

Def: The XOR operation is defined as $\{0,1\} \times \{0,1\} \to \{0,1\}$

$0 \oplus 1 := 1$

$1 \oplus 0 := 1$ (aka addition mod 2)

$0 \oplus 0 := 0$

$1 \oplus 1 := 0$

Extend this to binary strings by right-aligning them and applying XOR column-wise.

(If one string is shorter than the other, then pad it with 0s on the left.)

E.g.

$$\begin{array}{r} 111_2 \\ 01_2 \\ \oplus\ 11011_2 \\ \hline 11101_2 \end{array}$$

Rmks

- XOR is commutative: $x \oplus y = y \oplus x$
- XOR is associative: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
- $x \oplus x = 0$ for every x.

Rmk: Two binary strings are considered equal if they are the same up to an initial sequence of zeroes: $01_2 = 1_2 = 001_2$

** Some nim games (Nim sum)

- $(3,3,5)$: convert each pile size to a binary number: write as the sum of distinct powers of 2, and put a 1 for every power that appears, and a 0 for every power that doesn't appear.

$3 = 2 + 1 = 2^1 + 2^0 = 11_2$

$\quad\quad\quad\quad\ 1 \quad 1$

$$5 = 4 + 1 = 2^2 + 2^0 = 101_2$$

$$\quad 1 \quad 0 \quad 1$$

Nim sum of $(3,3,5)$ is also denoted

$$3 \oplus 3 \oplus 5 = \underbrace{11_2 \oplus 11_2}_{=0} \oplus 101_2 = 101_2$$

$(2,3,1)$

$$2 \oplus 3 \oplus 1 = 10_2 \oplus 11_2 \oplus 1_2 = \begin{array}{r} 10 \\ 11 \\ \oplus\ 01 \\ \hline 00_2 \end{array} = 0_2.$$

(n,n) and $n \oplus n = 0_2$

** Def: Let $(n_1, \cdots, n_k)$ be a game state for nim. Its nim-sum is $n_1 \oplus \cdots \oplus n_k$. That is, the XOR of the binary representations of $n_1, n_2, \cdots, n_k$.

** Theorem: A game position in nim is a:

P position iff its nim-sum is 0.

N position iff its nim-sum is non-zero.

** Example: $(4, 5, 6, 13)$

$$\left.\begin{array}{r} 100_2 \\ 101_2 \\ 110_2 \\ 1101_2 \\ \hline 1010_2 \end{array}\right\} \Rightarrow \text{winning position (by thm)} \ (N)$$

Theorem + our knowledge of N/P labelling tells us that there must be a move that takes us to a P position, i.e one whose nim-sum is zero.

** Example: $(4, 5, 6, 13) \overset{?}{\to} (4,5,6,7)$

$$\begin{array}{r} 100_2 \\ 101_2 \\ 110_2 \\ 1101_2 \\ \hline 1010_2 \end{array} \qquad \begin{array}{r} 100_2 \\ 101_2 \\ 110_2 \\ 111_2 \\ \hline 000_2 \end{array}$$

A winning move is one that keeps an even number of 1s in each column $\iff$ XOR is zero

** Example $(7, 4, 5) \to (1,4,5)$, $(7,2,5)$, $(7,4,3)$

$$\begin{array}{r} 111_2 \\ 100_2 \\ \oplus\ 101_2 \\ \hline 110_2 \end{array}$$

Observe: A state is "N" iff at least one of the columns has an odd number of 1s.

Strategy: Consider the leftmost column w/ an odd number of 1s. Choose any one number that has a 1 in this column.

$$\begin{array}{r} \boxed{111_2}\ n \\ 100_2 \\ \oplus\ 101_2 \\ \hline \boxed{110_2}\ s \end{array}$$

<u>Strategy</u>: Take this number, say n.
Let s be the nim-sum
Take $n \oplus s$. Convert to decimal, and replace n by this new number.

$$\begin{array}{r} 111_2 \\ \oplus\ 110_2 \\ \hline 001_2 \end{array} = 1 \text{ (in decimal)} \Rightarrow (7,4,5) \to (1,4,5)$$

or,
$$\begin{array}{r} 100 \\ \oplus\ 110_2 \\ \hline 010_2 \end{array} = 2 \Rightarrow (7,4,5) \to (7,2,5)$$

or
$$\begin{array}{r} 101_2 \\ \oplus\ 110_2 \\ \hline 011_2 \end{array} = 3 \Rightarrow (7,4,5) \to (7,4,3)$$