MATH 2301

* Nim strategy

** Example

(1, 2, 5, 7)

N-position, according to the theorem.

$$
\begin{array}{r}
1_2 \\
10_2 \\
101_2 \\
\oplus \; 111_2 \\
\hline
001_2
\end{array} \neq 0
$$

** Theorem

A position in nim is an "N" position iff its nim-sum is non-zero. It is a "P" position iff its nim-sum is zero.

** Proof sketch

We need to show the following:

1) Any move from a position with sum $= 0$ lands us in a position with nonzero sum.

✓ 2) From a position with sum $\neq 0$, there is at least one move to a position with sum $= 0$.

$$
\begin{array}{rl}
1_2 & \\
10_2 & \\
101_2 & n \\
\oplus \; 111_2 & \\
\hline
001_2 & s
\end{array}
$$

Steps to achieve (2):

1) Look at first column from the left with an odd number of 1s.

2) Choose a pile that has a 1 in that column.

3) Let n be chosen pile, and s be the nim-sum.

4) Take $n \oplus s$, and replace n with $n \oplus s$

$$\left.\begin{array}{r} 101_2 \\ \oplus\ 001_2 \\ \hline 100_2 \end{array}\right\}$$ the move $(1,2,4,7)$ has nim-sum 0.

More generally: $(x_1, \dots, x_k)$ be a nim config.

$$S = x_1 \oplus \cdots \oplus x_k$$

Suppose $s > 0$.

Follow steps ① & ②. Suppose that x_m is a pile size that has a 1 in the leftmost column with an odd number of 1s.

Make the following move: Change x_m to $(x_m \oplus s)$

Why is this a valid move? Is the new nim-sum 0?

$$\begin{array}{r} 1_2 \\ 10_2 \\ 101_2 \\ \oplus\ 111_2 \\ \hline 001_2 \end{array} \quad \rightsquigarrow \quad \begin{array}{r} 1_2 \\ 10_2 \\ 100_2 \\ \oplus\ 111_2 \\ \hline 000_2 \end{array}$$

** Prop: The new nim-sum is zero.

Pf: $x_1 \oplus \cdots \oplus x_m \oplus \cdots \oplus x_k = s$ (old eqn)

$x_1 \oplus \cdots \oplus (x_m \oplus s) \oplus \cdots \oplus x_k = ?$ (new eqn)

commutativity of $\oplus$ $\rightarrow$ $= s \oplus (x_1 \oplus \cdots \oplus x_k) = s \oplus s = 0$

** **Prop** Under the previous assumptions, $(x_m \oplus s) < x_m$.

Pf:

Note: x_m has a 1 in the first column from the left in which s has a 1.

$$\begin{array}{rr} & 1_2 \\ & 10_2 \\ & 101_2 \\ \oplus & 111_2 \\ \hline & 001_2 \end{array}$$

$x_m \oplus s$ in binary looks like:

$$\begin{array}{rl} & * \; * \cdots * \; 1 \; * \; * \; * \cdots * \\ \oplus & 0 \; 0 \; 0 \cdots 0 \; 1 \; * \; * \; * \cdots * \\ \hline & \underbrace{* \; * \cdots *}_{\text{same as } x_m} \; 0 \; \underbrace{* \; * \cdots *}_{\text{something}} \end{array}$$

$\leftarrow x_m$, $\leftarrow s$ } $* =$ unknown.

$$\begin{array}{l} q \; \cdots \; p{+}1 \; p \; \cdots \; 1 \; 0 \\ * \; * \cdots * \; 1 \; * \; * \; * \cdots * \\ 0 \; 0 \; 0 \cdots 0 \; 1 \; * \; * \; * \cdots * \\ \hline * \; * \cdots * \; 0 \; * \; * \cdots * \end{array}$$

$\rightsquigarrow x_m$ contains 2^p

$\rightsquigarrow (x_m \oplus s)$ does not contain 2^p, but all higher places equal x_m.

This implies that $(x_m \oplus s) < x_m$, because the largest possible number in the second part of $(x_m \oplus s)$ (after the 0) can be $111\cdots1_2 = (1+2+2^2+\cdots+2^{p-1})$. This sum equals $(2^p - 1) < 2^p$.

$\Rightarrow$ Changing x_m to $x_m \oplus s$ is a valid nim move.

Part (1) of the proof:

Suppose $(x_1, \dots, x_k)$ is a game position with
$x_1 \oplus \cdots \oplus x_k = 0$.

Suppose we make a move in x_m, changing it to x_m'.

New nim-sum:

$x_1 \oplus x_2 \oplus \cdots x_m' \oplus \cdots \oplus x_k = s$.

Consider $s \oplus 0 = s = (x_1 \oplus \cdots \oplus x_m' \oplus \cdots \oplus x_k) \oplus (x_1 \oplus \cdots \oplus x_m \oplus \cdots \oplus x_k)$

$s = x_m' \oplus x_m$. (everything else cancels).

$$\left.\begin{array}{rl} x_m = & * * \cdots * * \\ x_m' = & \underline{* * \cdots * *} \oplus \\ & * * \cdots * * = s \end{array}\right\}$$

$s = 0$ if and only if $x_m' = x_m$ binary digit by digit.

So $s = 0$ iff $x_m' = x_m$.

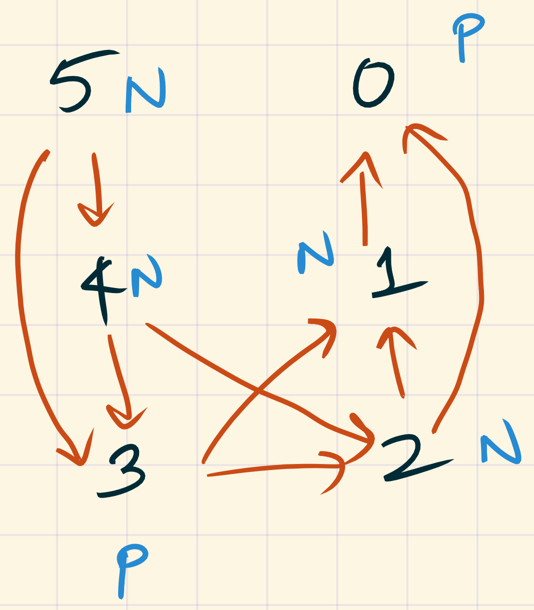
But $x_m' < x_m$ (we made a move)

$\Rightarrow s \neq 0$

** Grundy labelling

Let G be any impartial combinatorial game.

Eg. : n = 5, subtraction game with S = {1,2}



Grundy labelling = more sophisticated labelling of the game graph.