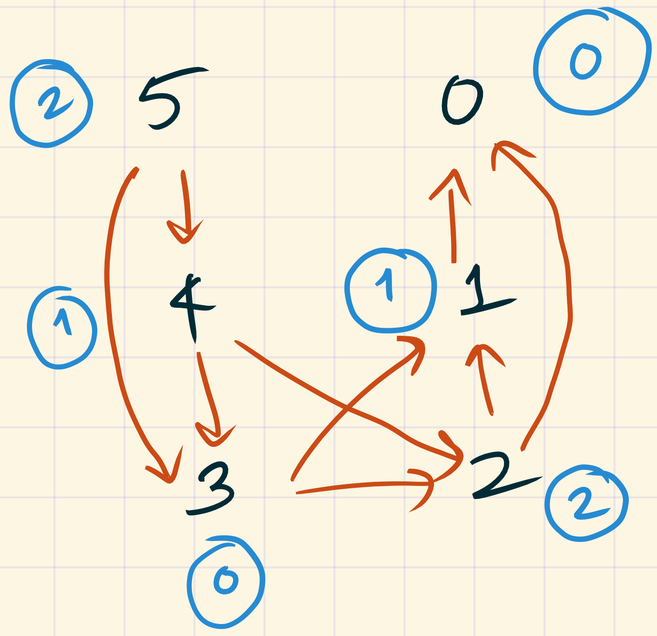
MATH 2301

* Grundy labelling

Eg: $n = 5$, subtraction game with $S = \{1,2\}$



1) Terminal position $\mapsto 0$
2) Any other position $\mapsto$ "mex" of labels of anything it points to.

"minimum excluded"

* **Def**: The Grundy labelling of a game graph is defined as follows:
  1) Terminal positions (those without outgoing edges) are labelled 0
  2) A position with outgoing arrows to positions labelled $a_1, a_2, \dots, a_k$ is labelled by the $\text{mex}\{a_1, \dots, a_k\}$ = the minimum integer $m \geq 0$ which is not in the set $\{a_1, \dots, a_k\}$.

Prop: A position is a P position iff its Grundy label is 0

A position is an N position iff its Grundy label is non-zero.

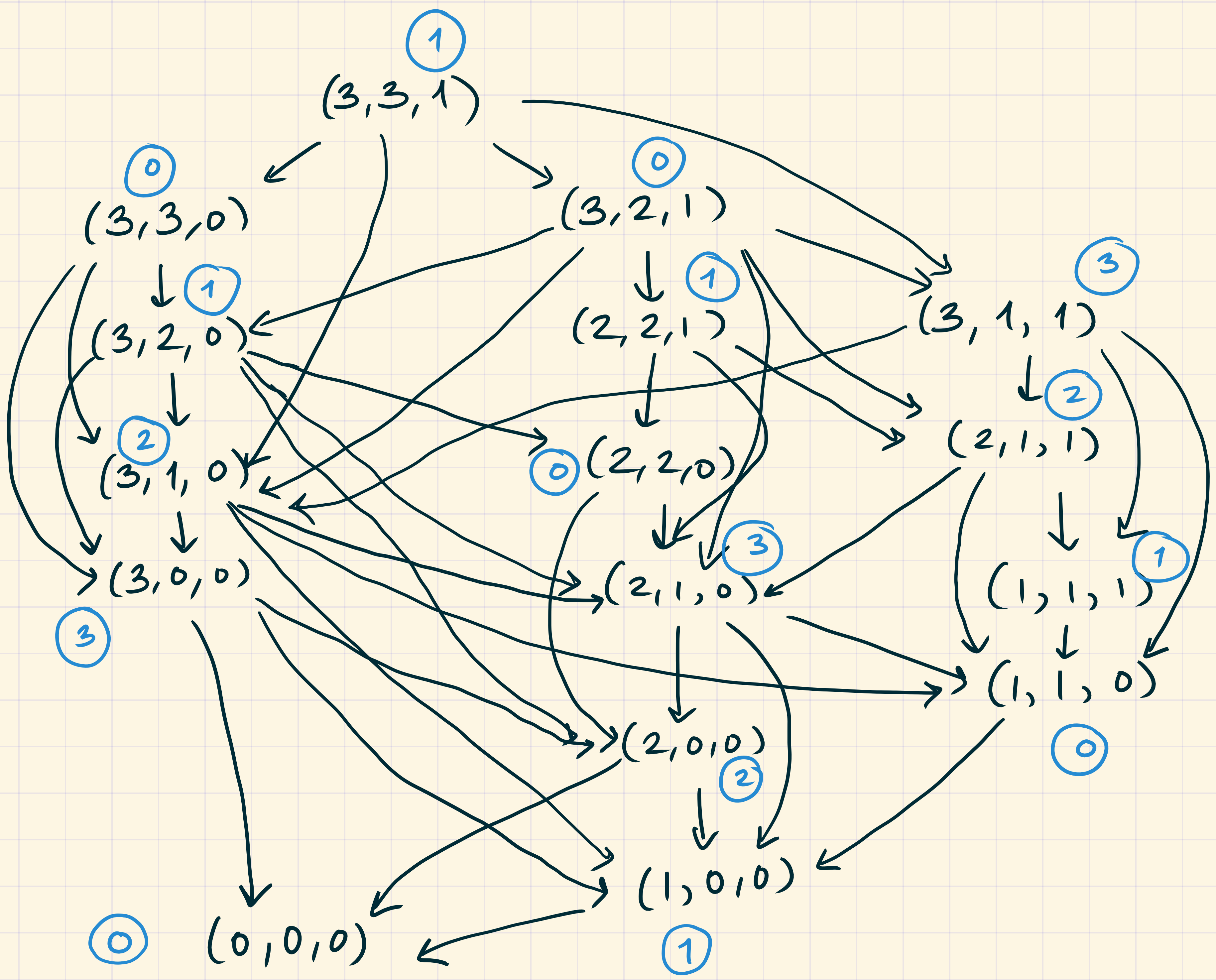
$\Rightarrow$ The Grundy label has more info than the N/P label.

Pf sketch.

1) Terminal positions are P w/ Grundy label 0.
2) Any other position is N iff it points to at least one P position (as per N/P convention), and labelled 0 iff it only points to non-zero labels.

(& use induction.)

** Example : Nim with (3,3,1)



Theorem: The Grundy label of a nim position $n_1, \dots, n_k$ is precisely $n_1 \oplus \cdots \oplus n_k$.

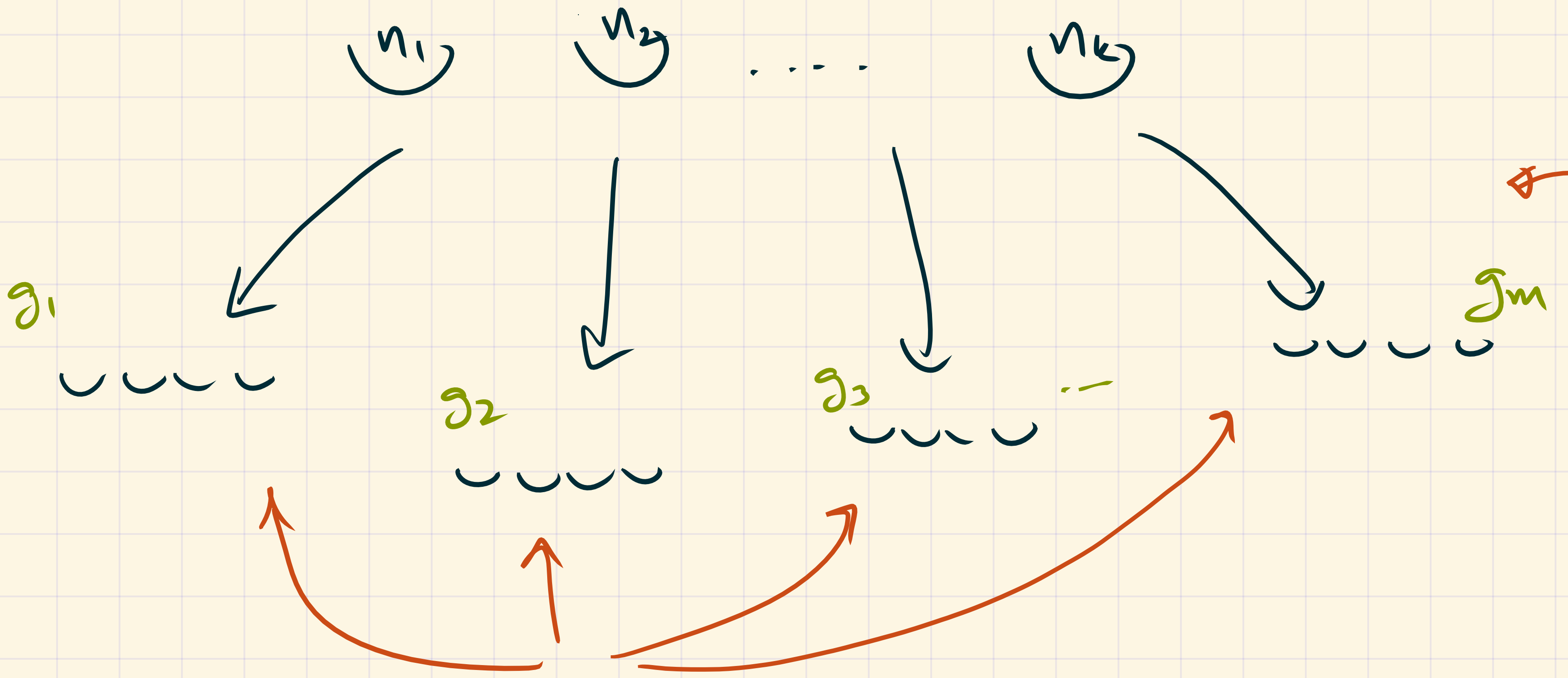
Proof (sketch)

initial number of berries

(By induction on the usual sum $n_1 + \cdots + n_k$).

1) If all piles have size 0, it is a terminal position. $\Rightarrow$ Grundy label of 0 = nim sum in this case.

2) Suppose they are not all size 0.



possible moves

By induction, we know that the Grundy label is just the nim sum, because the total number of berries has decreased.

Claim: $n_1 \oplus n_2 \oplus \cdots \oplus n_k$ = mex of all the Grundy labels $(g_1, \dots, g_m)$

$\Rightarrow$ nim-sum = Grundy label.

To show this, we show that:

1) $(n_1 \oplus n_2 \oplus \cdots \oplus n_k)$ is not reachable from $(n_1, \ldots, n_k)$. i.e, it is excluded.

2) Anything $<$ $(n_1 \oplus \cdots \oplus n_k)$ is reachable from $(n_1, \ldots, n_k)$. ie. it is the minimum excluded, or mex.

Let's prove (2):

Let $s = n_1 \oplus \cdots \oplus n_k$.

In binary:

$$\begin{array}{r} (n_1)_2 \\ (n_2)_2 \\ \vdots \\ \oplus\ (n_k)_2 \\ \hline 1 \ast \cdots\cdots \ast \end{array} \leftarrow s_2$$

$1 \ast \cdots \ast 1 \ast \cdots \ast \ast$ ← s_2

$\underbrace{1 \ast \cdots \ast}_{\text{same as } s} \underbrace{0}_{\text{first "1" from the left that becomes a } 0} \underbrace{\ast \cdots \ast \ast}_{\text{not determined.}}$ ← a binary number less than s.

Eg. If $s = 18$,

$s_2 = 1\ 0\ 0\ \underline{1}\ 0$

$(16_2) = \underbrace{1\ 0\ 0}_{\text{same as } s}\ \underbrace{0}_{\text{first flipped } 1}\ \underbrace{0}$

$(5)_2 = \underbrace{0}_{\text{first flipped } 1}\ \underbrace{0\ 1\ 0\ 1}$

To show that any $s' < s$ is reachable, we look at the nim sum again:

$$
\left.\begin{array}{r}
(n_1)_2 \\
(n_2)_2 \\
\vdots \\
\oplus \quad (n_k)_2 \\
\hline
1 * \cdots \quad *
\end{array}\right\}
$$

s' has some 1 flipped to 0, and some other changes afterwards.

This can be engineered using a single n_i that has a 1 in that position ...