

MATH 2301

* Theorem: The Grundy label of the nim position (n_1, \dots, n_k) is exactly $n_1 \oplus \dots \oplus n_k$.

** Pf sketch: We show the following:

1) A nim-sum of any $s' < s = n_1 \oplus \dots \oplus n_k$ is achievable by a single nim move.

2) A nim sum of $s = n_1 \oplus \dots \oplus n_k$ is not achievable by a single nim move.

① & ② show that $s = \text{mex}$ of the nim sums of all reachable positions.

By induction, this proves the theorem.

*** Pf sketch of 1

Consider some $s' < s = n_1 \oplus \dots \oplus n_k$.

Then look the leftmost column in $(s')_2$ that differs from s_2 .

Since $s' < s$, it must be the case that in that column, s has a 1 & s' has a 0.

E.g.	$s = 7$	1 1 1	$s = 7$	1 1 1	} $(s \oplus s')_2$
	$s' = 3$	0 1 1	$s' = 4$	1 0 0	
	$(s \oplus s')_2 = 1 0 0$			0 1 1	

Therefore $s \oplus s'$ (in binary) begins at this column (the leftmost column where s & s' differ).

Since s also has a 1 in that column,

there is some m such that n_m also has a 1 in the same column.

Make the following move:

$$(n_1, n_2, \dots, n_k) \rightsquigarrow (n_1, \dots, n_{m-1}, n'_m, n_{m+1}, \dots, n_k)$$

$$n'_m = n_m \oplus s \oplus s'$$

(Check: $n'_m < n_m$)

$$\text{New nim sum} = \underbrace{(n_1 \oplus \dots \oplus n_{m-1} \oplus n_m \oplus n_{m+1} \oplus \dots \oplus n_k)}_{= s} \oplus s \oplus s' = s'$$

*** Pf sketch of 2

We show that if we make any move, say $n_m \mapsto n'_m$, then the new nim sum cannot be s again.

Because if we had

$$s = n_1 \oplus n_2 \oplus \dots \oplus n_{m-1} \oplus n'_m \oplus n_{m+1} \oplus \dots \oplus n_k,$$

then

$$\left. \begin{aligned} n_1 \oplus n_2 \oplus \dots \oplus n_m \oplus \dots \oplus n_k &= \\ n_1 \oplus n_2 \oplus \dots \oplus n'_m \oplus \dots \oplus n_k & \end{aligned} \right\} \text{Add everything except } n_m \text{ to both sides, and they cancel}$$

$\Rightarrow n_m = n'_m$, a contradiction.

* Sum of games

Let G & H be (impartial, combinatorial) games. Then $G+H$ is the game whose state is the union of the states of G & H . That is, we play G & H "in parallel", with the following rules.

- 1) To make a move: either make a single move in G or make a single move in H .
- 2) You lose if there are no moves possible in either of the two games.

** Basic example: nim again.

Notation

Say that $*s$ is the nim game with a single pile of size s .

A nim game (n_1, \dots, n_k) is just the sum $(*n_1) + (*n_2) + \dots + (*n_k)$ as a game.

* Other examples: $(3 \times 4 \text{ chomp}) + (\text{Grundy's game with position } 6)$
etc

Questions: How do you know if $G+H$ is N or P?

Given the N/P values of G and H , what can we say about $G+H$?

Given the Grundy values of G & H , what is the Grundy value of $G+H$?

Answers (for nim)

If $(n_1, \dots, n_k) \stackrel{G}{=} G$ & $(m_1, \dots, m_\ell) \stackrel{H}{=} H$ are nim games, we know their Grundy values: $g = (n_1 \oplus \dots \oplus n_k)$ and $h = (m_1 \oplus \dots \oplus m_\ell)$ from Thm.

$G+H$ is just a bigger nim game:

$(n_1, \dots, n_k, m_1, \dots, m_\ell)$

Its Grundy value is $n_1 \oplus \dots \oplus n_k \oplus m_1 \oplus \dots \oplus m_\ell = g \oplus h$.

** Theorem: The Grundy label of $G+H$, given that g & h are the Grundy labels of G & H , is just $g \oplus h$.

Pf: Skip, but similar to the proof from the beginning of class.

** Remarks (consequences)

- 1) If G & H are both P positions, then $G+H$ is a P position.
[P positions have a Grundy value of 0]
- 2) If H is a P position, and G is any game, then the Grundy value of $G+H$ is the same as that of G .
- 3) If G is any game, then $G+G$ is a P position.
- 4) Warning: Simply knowing that G & H are both N positions does not tell us whether $G+H$ is N or P.

E.g. $G = (1, 3)$ nim } both N positions
 $H = (1, 4, 7)$ nim }

nim values are $g = \begin{array}{r} 11_2 \\ 1_2 \\ \hline (10)_2 = 2 \end{array} \quad g = 2$

$$h = \begin{array}{r} 111_2 \\ 101_2 \\ \hline 1_2 \\ 11_2 = 3 \end{array} \quad h = 3$$

$(G+H)$ has a nim/Grundy value of $1 = g \oplus h$
 $(G+G)$ has a nim/Grundy value of 0