MATH 2301

* **Theorem**: The Grundy label of the nim position $(n_1, \dots, n_k)$ is exactly $n_1 \oplus \dots \oplus n_k$.

** **Pf sketch**: We show the following:

1) A nim-sum of any $s' < s = n_1 \oplus \dots \oplus n_k$ is achievable by a single nim move.

2) A nim sum of $s = n_1 \oplus \dots \oplus n_k$ is not achievable by a single nim move.

1 & 2 show that $s =$ mex of the nim sums of all reachable positions.
By induction, this proves the theorem.

*** **Pf sketch of 1**

Consider some $s' < s = n_1 \oplus \dots \oplus n_k$.
Then look the leftmost column in $(s')_2$ that differs from s_2.

Since $s' < s$, it must be the case that in that column, s has a 1 & s' has a 0.

E.g.

$s = 7$	1 1 1		$s = 7$	1 1 1	$(s \oplus s')_2$
$s' = 3$	0 1 1		$s' = 4$	1 0 0	0 1 1

$(s \oplus s')_2 =$ 1 0 0

Therefore $s \oplus s'$ (in binary) begins at this column (the leftmost column where s & s' differ).

Since s also has a 1 in that column,

there is some m such that n_m also has a 1 in the same column.

Make the following move:

$(n_1, n_2, \dots, n_k) \rightsquigarrow (n_1, \dots, n_{m-1}, n_m', n_{m+1}, \dots, n_k)$

$n_m' = n_m \oplus s \oplus s'$

(Check: $n_m' < n_m$)

New nim sum $= (\overbrace{n_1 \oplus \cdots \oplus n_{m-1} \oplus n_m \oplus n_{m+1} \cdots \oplus n_k}^{= s}) \oplus s \oplus s'$

$= s'$

*** Pf sketch of 2

We show that if we make any move, say $n_m \mapsto n_m'$, then the new nim sum cannot be s again.

Because if we had

$s = n_1 \oplus n_2 \oplus \cdots \oplus n_{m-1} \oplus n_m' \oplus n_{m+1} \oplus \cdots \oplus n_k$,

then

$$\left.\begin{aligned} &n_1 \oplus n_2 \oplus \cdots \oplus n_m \oplus \cdots \oplus n_k = \\ &n_1 \oplus n_2 \oplus \cdots \oplus n_m' \oplus \cdots \oplus n_k \end{aligned}\right\}$$

Add everything except n_m to both sides, and they cancel

$\Rightarrow n_m = n_m'$, a contradiction.

* Sum of games

Let G & H be (impartial, combinatorial) games. Then G+H is the game whose state is the union of the states of G & H. That is, we play G & H "in parallel", with the following rules.

1) To make a move: either make a single move in G or make a single move in H.

2) You lose if there are no moves possible in either of the two games.

** Basic example: nim again.

Notation

Say that $*s$ is the nim game with a single pile of size s.

A nim game $(n_1, \cdots, n_k)$ is just the sum $(*n_1) + (*n_2) + \cdots + (*n_k)$ ⇝ As a game.

* Other examples: (3×4 chomp) + (Grundy's game with position 6)

etc

Questions: How do you know if $G+H$ is N or P?

Given the N/P values of G and H, what can we say about $G+H$?

Given the Grundy values of G & H, what is the Grundy value of $G+H$?

Answers (for nim)

If $\overbrace{(n_1, \dots, n_k)}^{=G}$ & $\overbrace{(m_1, \dots, m_\ell)}^{=H}$ are nim games, we know their Grundy values:

$g = (n_1 \oplus \cdots \oplus n_k)$ and $\underbrace{(m_1 \oplus \cdots \oplus m_\ell)}_{h=}$ from Thm.

$G+H$ is just a bigger nim game:

$(n_1, \dots, n_k, m_1, \dots, m_\ell)$

Its Grundy value is $n_1 \oplus \cdots \oplus n_k \oplus m_1 \oplus \cdots \oplus m_\ell$

$= g \oplus h.$

** Theorem: The Grundy label of $G+H$, given that g & h are the Grundy labels of G & H, is just $g \oplus h$.

Pf: Skip, but similar to the proof from the beginning of class.

** **Remarks** (consequences)

1) If G & H are both P positions, then G + H is a P position.
[P positions have a Grundy value of 0]

2) If H is a P position, and G is any game, then the Grundy value of G+H is the same as that of G.

3) If G is any game, then G + G is a P position.

4) **Warning**: Simply knowing that G & H are both N positions does not tell us whether G + H is N or P.

E.g. $G = (1, 3)$ nim, $H = (1, 4, 7)$ nim $\Big\}$ both N positions

nim values are

$$g = \begin{array}{r} 11_2 \\ 1_2 \\ \hline (10)_2 = 2 \end{array} \qquad g = 2$$

$$h = \begin{array}{r} 111_2 \\ 101_2 \\ 1_2 \\ \hline 11_2 = 3 \end{array} \qquad h = 3$$

$(G+H)$ has a nim/Grundy value of $1 = g \oplus h$
$(G+G)$ has a nim/Grundy value of 0