MATH 2301

* Yesterday: sums of games

We saw that all P-position games "behave like 0".
(in terms of Grundy labelling)

* Equivalence of games

** Def: We say that G & G' are equivalent if for any game H, the two games $G+H$ and $G'+H$ are either both N, or both P.

In this case, we say $G \sim G'$.
It can be shown that this is an equivalence relation.

** Rmk: If $G \sim G'$, then G is an N position iff G' is an N-position.
[see this by adding any terminal game position "aka the empty game" to both sides.]

* Warning. If G & G' are both N, then it need not be the case that they are equivalent.

** Prop: $*m$ & $*n$ are not equivalent, if $m \neq n$.
Pf: To see this, add $*m$ to both sides

$*m + *m$ is a P position
$*m + *n$ is an N position if $m \neq n$.

$\Rightarrow$ There are at least as many equivalence classes as the natural numbers.

** Prop: Let L be a P position, and G any game. Then $G \sim G+L$.

Pf sketch:
show: If $G+H$ is N, then $G+L+H$ is also N (and similarly if $G+H$ is P).

$$\begin{array}{c}(G+H)+L \\ \shortparallel \\ G+L+H\end{array}$$

Strategy to win $(G+H+L)$:
1) Make a winning move in $G+H$.
2) If opponent plays in L, you respond in L with a good move.
3) If opponent plays in $G+H$, respond in $G+H$.

[Similarly, you can show that if $G+H$ is P, then $G+H+L$ is P.]

** Cor : If L & L' are two P-position games, then $L \sim L'$.

Pf : Note that $L+L'$ is also a P-position. (from last time)

By prop, we have $\underline{\underline{L}} \sim L+(L+L')$

$L+(L+L') = (L+L)+L' \sim \underline{\underline{L'}}$ by proposition.

$\uparrow$ is also a P-position!

$\Rightarrow L \sim L'$.

---

* Prop : If $G \sim G'$, then $G+G'$ is a P-position.

Pf : Suppose $G \sim G'$, then $G+G'$ and $G'+G'$ have the same outcome (N/P).

But $G'+G'$ is P, so $G+G'$ is also P.

* Thm : If $(G+G')$ is a P-position, then $G \sim G'$.

Pf : We have to show that $G \sim G'$

We know: $G \sim G+(G+G')$ [$G+G'$ is P]

$= (G+G)+G'$

$\sim G'$ [$G+G$ is P]

** Prop : A nim game $(n_1, \dots, n_k)$ is equivalent to the game $*s$, where $s = n_1 \oplus \dots \oplus n_k$.

Pf : Let $G = (n_1, \dots, n_k)$ and $H = *s$

To show $G \sim H$, we just have to check that $G+H$ is P.

Because the nim value / Grundy value of $G+H$ is $(n_1 \oplus \dots \oplus n_k) \oplus s = s \oplus s = 0$

$\Rightarrow$ Every nim game is equivalent to a single-pile nim game.

* Theorem [Sprague-Grundy] :

Every impartial combinatorial game G is equivalent to $*g$, where g = Grundy value of G.

Pf Let $G' = *g$. We want to show $G \sim G'$

Let's instead show that $G+G'$ is a P position. We compute the Grundy value of $G+G'$: this is again just the nim sum of the Grundy values of G & G', which is $g \oplus g = 0$.