MATH 2301

01 Aug 2022

* Office hour: Wednesdays 3pm–4pm (HN 4.84) (or 4th floor meeting room)
  Zoom (by appointment)

* Assignment 1 due this Friday 05 August (Gradescope)

* Workshops begin this week! (HN 4.41?)

Last time: graphs & adjacency matrices (more about this next week)

Equivalence relations.

Def: A relation R on a set S is called an equivalence relation if it is
① reflexive, ② symmetric, and ③ transitive.

Rmk: Equivalence relations give us a more general/looser notion of two objects being similar/equal/equivalent.

Examples / non-examples

① R a relation on the set of integers $\mathbb{Z}$

$R = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid x+y \text{ is even}\}$

ⓐ Reflexivity: yes because $x+x = 2x$ is always even for $x \in \mathbb{Z}$

ⓑ Symmetry: yes : if $(x,y) \in R$ then $x+y = y+x$ is even. So $(y,x) \in R$.

ⓒ Transitivity: yes if $(x,y) \in R$ and $(y,z) \in R$. then $x+y$ is even & $y+z$ is even.

Argument #1

$\Rightarrow$ x, y are either both odd / both even

y, z are either both odd / both even

$\Rightarrow$ x, z are either both odd / both even

$\Rightarrow$ their sum is even.

Argument #2

Add up $(x,y), (y,z)$: get $x+2y+z$

We know: $(x+y)+(y+z)$ is even

$(x+z)$ = an even number $- 2y$.

This must be even.

$\Rightarrow$ R is an equivalence relation

② $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x+y \text{ is odd}\}$

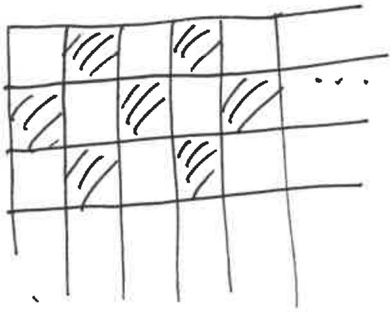
ⓐ Reflexivity: No! $x+x$ is always even.

(Not an equivalence relation!)

ⓑ Symmetric: Yes

ⓒ Transitive: ?

Exercises.

③ 

Chessboard (8×8)

S = set of squares on this chessboard.

$R = \{(s,t) \in S \times S \mid t$ is reachable from s by a sequence of bishop moves.$\}$

ⓐ Reflexive: Yes! (Don't move the bishop, or move back & forth)

ⓑ Symmetric: Yes! (Go backwards)

ⓒ Transitive: Yes! If $(s,t) \in R$ and $(t,u) \in R$ then t is reachable from s & u is reachable from t.

$\Rightarrow$ u is also reachable from s

$\Rightarrow$ $(s,u) \in R$.

So R is an equivalence relation.

(4) $S =$ set of squares on a chessboard

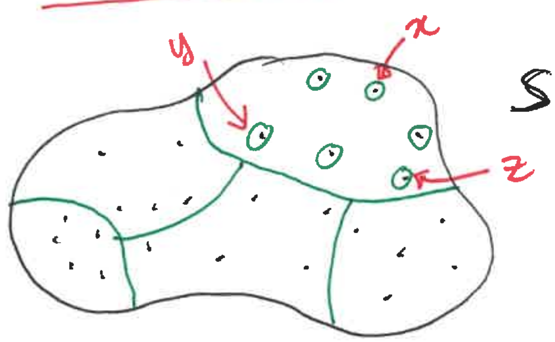
$R = \{(s,t) \in S \times S \mid t$ is reachable from s by at most one bishop move.$\}$

Exercise!

Notation: If R is an equivalence relation, and $(s,t) \in R$, then we often write

$s \sim_R t$ or simply $s \sim t$ if the relation is clear from context.

Equivalence classes of an equivalence relation



S, R is some equivalence relation on S.

Let $x \in S$.
Consider $\{ t \in S \mid (x,t) \in R\}$

~~$\{y \in S \mid (x,y) \in R\}$~~

Let y & z be two elements of this subset.
We know: $(x,y) \in R$ and $(x,z) \in R$.
By symmetry, $(y,x) \in R$.
By transitivity, $(y,z) \in R$.

Def: Let R be an equivalence relation on a set S.

Let $x \in S$. The equivalence class of x is:

$$\{ t \in S \mid (x,t) \in R \} = [x]_R = [x]$$

This is a subset of S, and it is denoted $[x]$.

Proposition

Let R be an equivalence relation on S.

Let $x \in S$

1. If $y, z \in [x]$ then $y \sim_R z$.

2. If $y \notin [x]$ and $z \in [x]$, then $y \not\sim_R z$