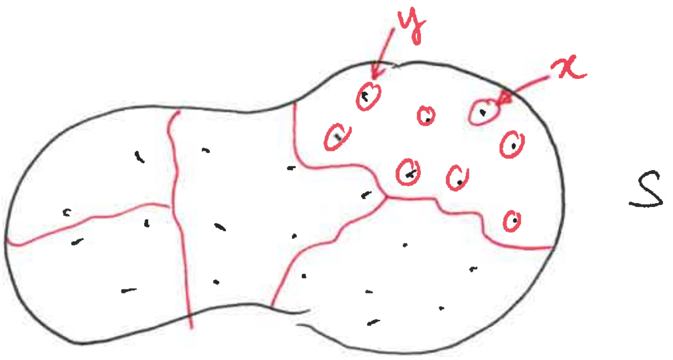
MATH 2301 — 03 August 2022

* Last time: Equivalence classes

Let R be an equivalence relation on a set S.
Let $x \in S$.
The equivalence class of x, denoted $[x]_R = [x]$
$= \{y \in S \mid (x,y) \in R\}$

Proposition

1) If $y, z \in [x]$, then $y \sim_R z$, i.e. $(y,z) \in R$.
2) If $y \notin [x]$ and $z \in [x]$, then $y \not\sim_R z$, i.e. $(y,z) \notin R$.
3) If $y \in [x]$, then $x \in [y]$, and in fact, $[x] = [y]$
4) If E_1 and E_2 are two equivalence classes, then either $E_1 = E_2$, or $E_1 \cap E_2 = \phi$.



Pf sketch:

1) If $y, z \in [x]$, then $(x,y) \in R$ & $(x,z) \in R$
By symmetry + transitivity, $(y,z) \in R$.

2) $y \notin [x]$, $z \in [x]$ Want to show that $y \not\sim z$.

If $y \sim z$, we'd have $(y,z) \in R \Leftrightarrow (z,y) \in R$.

Since $z \in [x]$, we have $(x,z) \in R$

$\Rightarrow (x,y) \in R$ by transitivity ← definitely false, because $y \notin [x]$

$\Rightarrow y \not\sim z$.

3) If $y \in [x] \Rightarrow (x,y) \in R$

By symmetry, $(y,x) \in R \Rightarrow y \in [x]$

Exercise : Use properties of R to show that $[x] = [y]$

4) Let E_1 & E_2 be equivalence classes.

If $E_1 \cap E_2 = \phi$ then we're done.

If $x \in E_1$ and $x \in E_2$, then ?

Suppose $E_1 = [y]$, $E_2 = [z]$

$\Rightarrow (y,x) \in R$ and $(z,x) \in R$

$\Rightarrow (y,z) \in R$, and $(z,y) \in R \Rightarrow y \in [z]$, $z \in [y] \Rightarrow [y] = [z]$

Notation

Let E be an equivalence class.
If $a \in E$, then a is called a representative of E, and $E = [a]$

Modular arithmetic

Example $S = \mathbb{Z}$ integers.

1) $R_2 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y$ is divisible by $2\}$

(an equivalence relation).

2) $R_{15} = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x-y$ is divisible by $15\}$

Question

Equivalence classes of R_2 ?

Note: If x, y are ~~ar~~ both even then $x \sim y$

If x, y are both odd, then ~~xxx~~ $x \sim y$

If one even & the ~~od~~ other is odd, then $x \not\sim y$

$\Rightarrow$ $[2] = \{ \ldots, -4, -2, 0, 2, 4, 6, \ldots \}$

$\quad \| $

$[-56] = [6] = [0] \ \ldots$

$[17] = \{ \ldots, -3, -1, 1, 3, 5, \ldots \}$

$\quad \|$

$[-57] = [1] = [13] = \ldots$

The set of equivalence classes of R_2

is $\{ [5], [22] \}$

odds evens

Equivalence classes of R_5?

(rem.4) $[9] = \{\ldots, -6, -1, 4, 9, 14, 19, \ldots\} = [4]$

(rem.0) $[0] = \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\} = [0]$

(rem.1) $[1] = \{\ldots, -9, -4, 1, 6, 11, \ldots\} = [1]$

(rem.2) $[7] = \{\ldots, -8, -3, 2, 7, 12, \ldots\} = [2]$

(rem.3) $[3] = \{\ldots -7, -2, 3, 8, 13, \ldots\} = [3]$

In general, you can write R_d for any positive integer d:

$\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid (x-y) \text{ is divisible by } d\}$

R_d will have exactly d equivalence classes.

The standard labelling is:

$$\underset{\substack{\| \\ [-5d]_d}}{[0]_d}, \quad [1]_d, \quad [2]_d, \quad \underset{\substack{\| \\ [3d+3]_d}}{[3]_d}, \quad \ldots \quad [d-1]_d.$$

* Modular addition.

Consider R_d.

Set $\mathbb{Z}_d :=$ set of equivalence classes of R_d

$$\mathbb{Z}_d = \{[0], [1], \ldots, [d-1]\}$$

We'll define an operation $+_d$ or $+$ on $\mathbb{Z}_d$

as follows:

$$[r] +_d [s] := [r+s]$$

Is this well-defined? → Next time.