MATH2301 Lecture 8

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* Last time:

- Graphs (directed, undirected, multi-)
- Matrices
- Adjacency Matrix

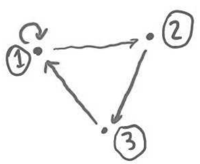
Questions:

1. Given two vertices, is there a path between them?
2. If so, how many paths are there?
3.

⋮

Reminder

Graph ⟶ Adjacency Matrix



$$\begin{array}{c|ccc} & & \text{to} & \\ \hline \text{from} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

What information does A^2 contain?

$A^2_{(i,j)}$ = dot product of ith row R_i and jth column C_j

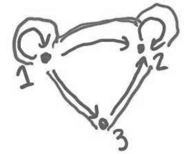
$$A^2_{(1,1)} = (\overset{1\to1\to1}{1\times 1}) + (\overset{1\to2\to1}{0\times 1}) + (\underset{1\to3\to1}{\overset{3\text{-}1 \quad 1\text{-}3}{1\times 0}}) = \underline{1} = \text{\# of length 2 paths from}$$

$$A^2_{(1,1)} = (1\times 1) + (0\times 1) + (\underset{1\to3\to1}{1\times 0}) = \underline{1} = \text{\# of length 2 paths from (1) to (1)}$$

\# of edges from 2-1; # edges 1-1; 2-1; from 1-2

Prop: The (i,j)th entry of A^2 gives the number of length two paths from i to j in the graph.

Why? Counts all the ways of going $i \to k \to j$ added up over all choices of k



$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

\# length 2 paths from 1 to 2

Theorem: The (i,j)th entry of $A^k = \overbrace{AA\ldots A}^{k}$ is exactly the number of length k paths from i to j.

Proof: If $k=1$, $A^k = A$, whose entries are the number of edges (length 1 paths) between vertices.

If $k \geq 1$, assume the result holds for $\underline{k-1}$, then

$$A^k_{(i,j)} = (A\,A^{k-1})_{(i,j)}$$

$$= \sum_{\ell \in V} \underset{\text{length 1 paths } i\to\ell}{A_{(i,\ell)}}\ \underset{\text{length } k-1 \text{ paths from } \ell\to j}{A^{k-1}_{(\ell,j)}}$$

So by induction, we're done.

Connectedness of graphs

Let $G = (\underline{V}, \underline{E})$ be a graph. Is there at least one path from the $\underline{i}$th vertex to the $\underline{j}$th vertex?

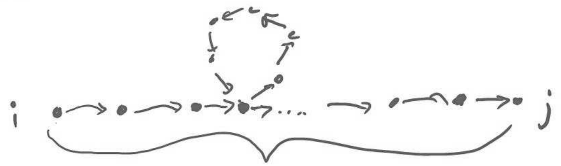
How can we check?

Check (i,j)th entry of A^k for all k

Check (i,j)th entry of A for all n

If any such entry is >0, then there is at least one path.

Follow-up question: If G has $\underline{n}$ vertices, how far might we have to travel?



If we allow length 0 path $i \to i$, then any vertices which can be connected can be connected by a path of length $\leq n-1$

If length 0 paths are excluded, best we can guarantee is $\leq n$.

So to check connectedness we can calculate

(allow length 0) $\left(\underline{I} + \underline{A} + \dots + \underline{A}^{n-1}\right)_{(i,j)}$ = paths $i \to j$ of length 0 to $n-1$

(disallow length 0) $\left(A + \dots + A^{n}\right)_{(i,j)}$ = paths $i \to j$ of length 1 to n.