MATH2301 Lecture 8

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* Last time:

- Graphs (directed, undirected, multi-)
- Matrices
- Adjacency Matrix

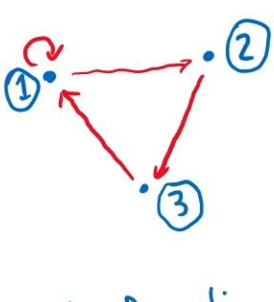
Questions:

1. Given two vertices, is there a path between them?
2. If so, how many paths are there?
3.

⋮

Reminder

Graph ⟶ Adjacency Matrix



$$\begin{array}{cc} & \text{to} \\ \text{from} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

What information does A^2 contain?

$A^2_{(i,j)}$ = dot product of ith row R_i and jth column C_j

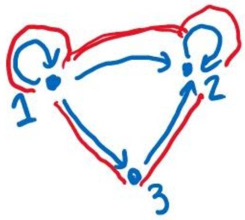
$$A^2_{(1,1)} = \overset{1\to1\to1}{(1 \times 1)} + \overset{1\to2\to1}{(0 \times 1)} + \underset{1\to3\to1}{(\overset{3\to1}{1} \times \overset{1\to3}{0})} = 1 =$$ # of length 2 paths from

$A^2_{(1,1)} = (1 \times 1) + (0 \times 1) + (1 \times 0) = 1 =$ # of length 2 paths from ① to ①

# of edges from 1-1 | # edges 1-1 | 2-1 | from 1-2 | $1 \to 3 \to 1$

Prop: The (i,j)th entry of A^2 gives the number of length two paths from i to j in the graph.

Why? Counts all the ways of going $i \to k \to j$ added up over all choices of k



$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

length 2 paths from 1 to 2

$$\overbrace{A\,A \dots A}^{k}$$

Theorem: The (i,j)th entry of A^k is exactly the number of length k paths from i to j.

Proof: If $k = 1$, $A^k = A$, whose entries are the number of edges (length 1 paths) between vertices.

If $k \geq 1$, assume the result holds for $\underline{k-1}$, then

$$A^k_{(i,j)} = (A\,A^{k-1})_{(i,j)}$$

$$= \sum_{\ell \in V} A_{(i,\ell)} A^{k-1}_{(\ell,j)}$$

length 1 paths $i \to \ell$

length $k-1$ paths from $\ell \to j$.

So by induction, we're done.

Connectedness of graphs

Let $G = (\underline{V}, \underline{E})$ be a graph. Is there at least one path from the $\underline{i}$th vertex to the $\underline{j}$th vertex?

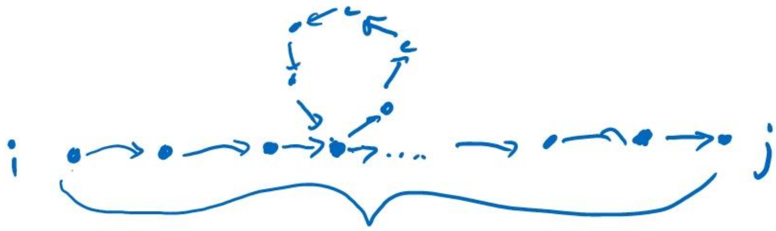
How can we check?

Check (i,j)th entry of A^k for all k

Check (i,j)th entry of A for all k

If any such entry is >0, then there is at least one path.

Follow-up question: If G has $\underline{n}$ vertices, how far might we have to travel?



If we allow length 0 path $i \to i$, then any vertices which can be connected can be connected by a path of length $\leq n-1$

If length 0 paths are excluded, best we can guarantee is $\leq n$.

So to check connectedness we can calculate

(allow length 0) $\quad (\underline{I} + \underline{A} + \dots + \underline{A}^{n-1})_{(i,j)} =$ paths $i \to j$ of length 0 to $n-1$

(disallow length 0) $\quad (A + \dots + A^n)_{(i,j)} =$ paths $i \to j$ of length 1 to n.