MATH 2301

15 Aug 2022

* Last time : Boolean product / addition

$\{0,1\}$ together with $\vee$ ("or"), $\wedge$ ("and")

$0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$, $\quad 0 \vee 0 = 0$

$0 \wedge 1 = 1 \wedge 0 = 0 \wedge 0 = 0$, $\quad 1 \wedge 1 = 1$

Boolean matrix product : multiply matrices with the same procedure, except:

replace "+" by "$\vee$"

replace "$\times$" by "$\wedge$"

Quick example : $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A$

$A^{*2} = 2^{nd}$ Boolean power

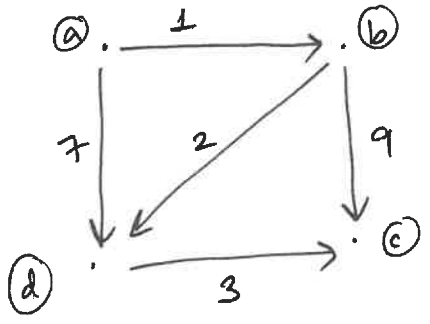
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Prop: Suppose you have a directed graph on n vertices, with an adjacency matrix A. ~~Then A^{*n} (n^{th} Boolean power) is the adjacency matrix of the transitive closure.~~

[Not quite correct → we need to take $A \vee A^{*2} \vee \dots \vee A^{*n}$ → we'll fix next time]

* Weighted directed graphs



Edges now have weights; which should be thought of as a "cost" or a "distance". We'll assume that all weights are non-negative numbers.

Question: What is the least-cost of travelling from one vertex to another?

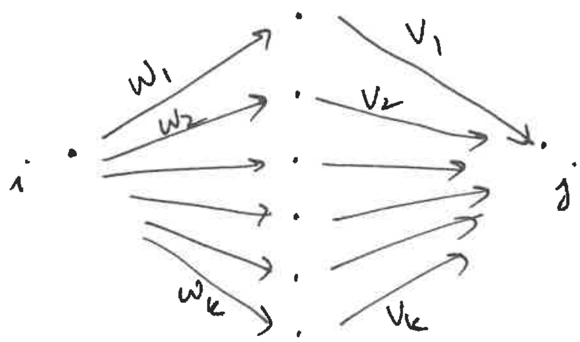
Weighted adjacency matrix:

~~Let~~ $W = \begin{bmatrix} 0 & 1 & \infty & 7 \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix}$

The $(i,j)^{th}$ entry is either:

$$\begin{cases} 0 & \text{if } i=j \\ \text{cost of edge} & \text{if } i \to j \\ \infty & \text{if there is no edge } i \to j \end{cases}$$

Example: To find the min. cost of a path from i to j of ≤ 2 edges, we look at:



min. cost

$= \min\{w_1+v_1, w_2+v_2, \ldots, w_k+v_k\}$

(all possible length ≤ 2 paths $i \to j$)

We'll do this using the "min-plus" product on weighted adjacency matrices.

Take W as in the previous example.

$W \circ W$ = notation for min-plus product

$$= \begin{bmatrix} \boxed{0 \quad 1 \quad \infty \quad 7} \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & \infty & 7 \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix} = \begin{bmatrix} \boxed{0} & \boxed{0} & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$\min\{0+0,\ 1+\infty,\ \infty+\infty,\ 7+\infty\}$$

$$\min\{0+1,\ 1+0,\ \infty+\infty,\ 7+\infty\}$$

"∞" is a symbol that denotes a cost that is higher than anything achievable.

$$\left.\begin{aligned} \Rightarrow \quad \infty + x &= \infty \\ \min\{\infty, x\} &= x \\ x + \infty &= \infty \\ \min\{x, \infty\} &= x \end{aligned}\right\} \text{ for every } x\text{, including } x = \infty$$

$$W \odot W = \begin{bmatrix} \min\{0,\infty,\infty,\infty\} & \min\{1,1,\infty,\infty\} & \min\{\infty,10,\infty,10\}, & \min\{7,3,\infty,7\} \\ \infty & 0 & 5 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 10 & 3 \\ \infty & 0 & 5 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix}$$

Prop: $W^{\odot k}$ = k^{th} min-plus power of W records the min costs of paths of length $\le k$ between any pair of vertices

Prop: Let G be a wtd graph with n vertices, and let W be its weighted adjacency matrix.
Then $W^{\odot(n-1)}$ records the min possible costs of paths of any length between any pair of vertices.
[The cost of going from i to i is zero by convention. ~~The~~ If there is a path from i to j and $i \neq j$, then there is always a path of length $\le n-1$.]