

- * Midterm exam → end of week 6
(Look on wattle → online exam)
(Syllabus → up until the end of week 5)

* Error from last time:

Transitive closure from Boolean powers

Prop If G is a graph w/ n vertices, and A is an adjacency matrix of G , the adjacency matrix of the transitive closure is:

$$A \vee A^{*2} \vee A^{*3} \vee \dots \vee A^{*n}$$

(entry-wise "or")

$$A \vee B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \vee \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} a_{11} \vee b_{11} & a_{12} \vee b_{12} & \dots \\ a_{21} \vee b_{21} & & \dots \\ & & \dots \end{pmatrix}$$

* Rmk: "Path" v.s "walk"

Sometimes, a "path" is a ~~walk~~ path through the graph that doesn't repeat vertices, while a walk can repeat vertices

BUT we will not put this restriction.

For me, both paths = walks can repeat vertices

* Partially ordered sets (posets)

Def: A ~~set~~ relation R on a set S is called a partial order if it is:

- ① reflexive,
- ② anti-symmetric, and
- ③ transitive.

** Notation: Let R be a partial order on S

- ① We'll say that S is a poset w.r.t R
- ② If $(a, b) \in R$, we'll say $a \preceq_R b$ or just $a \preceq b$

Examples

① $S = \mathbb{N}$

$$R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\}$$

Note: $x \preceq_R y$ if and only if $x \leq y$

② $S = \mathbb{N}$

$$R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \geq y\}$$

Note: $x \preceq_R y$ if and only if $x \geq y$

} check poset conditions

[In fact, these are total orders, meaning that given $a, b \in S$, either $a \preceq_R b$ or $b \preceq_R a$, or both]

③

Let A be a fixed set

E.g. $A = \{a, b, c\}$

$S = P(A) = \text{set of all subsets}$

E.g. $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \dots\}$

R is the "subset" relation
 " " } Check that this is a partial order
 $\{(X, Y) \in P(A) \times P(A) \mid X \subseteq Y\}$.

E.g. $X = \{a, b\}, Y = \{a, c\}$
 These are "not comparable", meaning that neither (X, Y) nor (Y, X) is in R .

E.g. $\{a\} \subseteq \{a, b\}, \{a\} \subseteq \{a, c\}$
 $\{a\} \preceq_R \{a, b\}, \{a\} \preceq_R \{a, c\}$.

④ $S = \{1, 2, 3, 4, 5, 6\}$

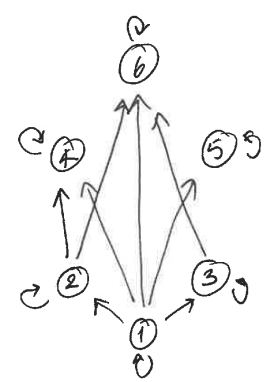
$R = \{(a, b) \in S \times S \mid a \text{ is a factor of } b\}$
 i.e. a/b
 reads "a divides b"
 i.e. a is a factor of b.

④

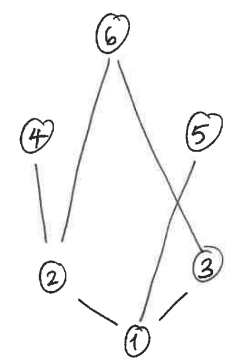
* Hasse diagram of a poset

(like the graph of the poset, but with redundant info removed)

E.g. $S = \{1, 2, 3, 4, 5, 6\}$
 $R = \{(a, b) \in S \times S \mid a \mid b\}$



Graph, where each non-loop edge "goes up"



Hasse diagram

Graph \rightarrow Hasse diagram

- ① Draw it so that all non-loop edges go upwards
- ② Delete all self-loops
- ③ Delete all edges implied by transitivity.
- ④ Remove all arrowheads, with the understanding that the lines are oriented upwards

[Reverse these steps to go from a Hasse diagram to the graph drawing]

E.g.

① $S = \{1, 2, 3, 4\}$

 \leq relation

④

③

②

①

(The Hasse diagram of any total order is a straight line.)

② $A = \{a, b, c\}$

 $S = \mathcal{P}(A)$, with the subset relation