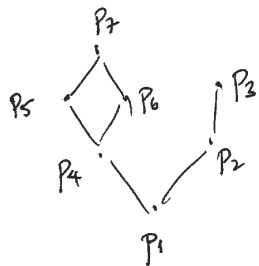
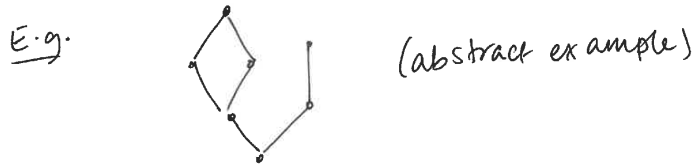


- * Posets [partially ordered sets]
- * Yesterday: Hasse diagrams
- * Today: Topological sort & incidence algebra.

Let (P, \leq) be a finite poset.

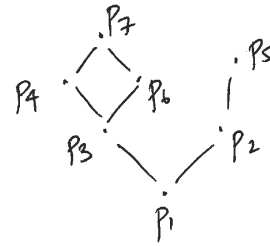


(topological sorting?)
 → label the vertices so that the numbers \uparrow increase as we go higher in Hasse diagram.

One possible example of a topological sorting

Informally: Start ^{by} labelling ~~the~~ one of the bottom-most elements. Next, in each step, choose a vertex such that everything \leq that vertex has already been labelled, and label it with the next number. Continue until everything has a label.

Another example



Def: A topological sorting of (P, \leq) is an ordered list of all the elements of P , say (p_1, p_2, \dots, p_n) , such that whenever $p_i \leq p_j$, then $i \leq j$.

Rmk: There are many possible such sortings.

Thm: If (P, \leq) is a finite poset, it has at least one topological sort.

Pf skipped, here's the idea.

We had an algorithm on the previous page to write down a labelling. We can check that

- ① the algorithm stops after a finite number of steps.
- ② it produces a topological sorting.

* Assorted definitions

Let (P, \leq) be a poset

- ① An element $x \in P$ is the maximum element of P if for every $y \in P$, we have $y \leq x$.

[Note: P may not have a maximum, but if it does, then it is unique.]

② An element $x \in P$ is the minimum element of P if for every $y \in P$, we have $x \leq y$.

[Note: Such an x is unique if it exists, but it may not exist.]

③ An element $x \in P$ is a maximal element of P if ~~for every~~ there does not exist $y \in P$ such that $y \neq x$ and $x \leq y$.

In other words, if there is a $y \in P$ such that $x \leq y$, then $y = x$.

[Note: Finite posets always have maximal elements, but they may not be unique.]

④ An element $x \in P$ is a minimal element of P if whenever there is a $y \in P$ such that $y \leq x$, then $y = x$.

[Note: Same as for maximal.]

Examples to play with

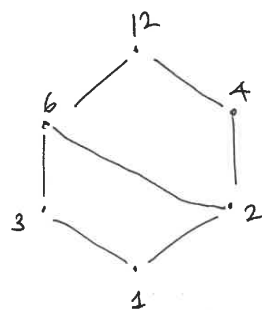


③

* Algebra of posets

Example $S = \{\text{factors of } 12\}$

Relation = "divisibility"



Def: Let (P, \leq) be a poset.

~~Let $x \leq y$~~ Let $x, y \in P$.

An interval I

The (closed) interval $[x, y]$ is the set

$$\{z \in P \mid x \leq z \text{ and } z \leq y\}.$$

E.g. In the poset above,

- $[1, 6] = \{1, 2, 3, 6\}$
- $[6, 1] = \emptyset$
- $[1, 2] = \{1, 2\}$
- $[3, 3] = \{3\}$

④