MATH 2301 — 18 Aug 2022

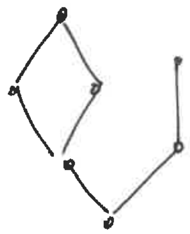
* Posets [partially ordered sets]

* Yesterday: Hasse diagrams

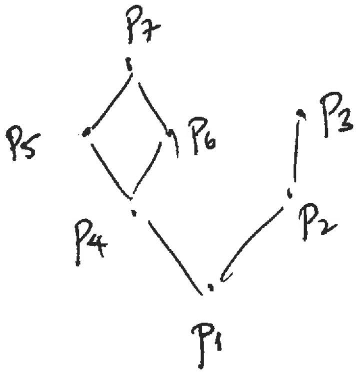
* Today: Topological sort & incidence algebra.

Let $(P, \preceq)$ be a finite poset.

E.g.



(abstract example)

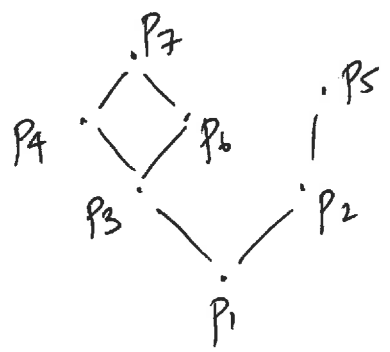


(topological sorting?)
→ label the vertices so that the numbers increase as we go higher in Hasse diagram.

One possible example of a topological sorting

Informally: Start by labelling ~~the~~ one of the bottom-most elements. Next, in each step, choose a vertex such that everything $\preceq$ that vertex has already been labelled, and label it with the next number. Continue until everything has a label.

Another example



Def: A topological sorting of $(P, \preceq)$ is an ordered list of all the elements of P, say $(p_1, p_2, \cdots, p_n)$, such that whenever $p_i \preceq_R p_j$, then $i \leq j$.

Rmk: There are many possible such sortings.

Thm: If $(P, \preceq)$ is a finite poset, it has at least one topological sort.

Pf skipped, here's the idea.
We had an algorithm on the previous page to write down a labelling. We can check that
① the algorithm stops after a finite number of steps.
② it produces a topological sorting.

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* Assorted definitions

Let $(P, \preceq)$ be a poset
① An element $x \in P$ is the maximum element of P if for every $y \in P$, we have $y \preceq x$.
[Note: P may not have a maximum, but if it does, then it is unique.]

2. An element $x \in P$ is the <u>minimum</u> element of P if for every $y \in P$, we have $x \preceq y$.

[<u>Note</u>: Such an x is unique if it exists, but it may not exist.]

3. An element $x \in P$ is a <u>maximal</u> element of P if there does not exist $y \in P$ such that $y \neq x$ and $x \preceq y$.

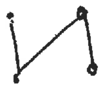
In other words, if there is a $y \in P$ such that $x \preceq y$, then $y = x$.

[<u>Note</u>: Finite posets always have maximal elements, but they may not be unique.]

4. An element $x \in P$ is a <u>minimal</u> element of P if whenever there is a $y \in P$ such that $y \preceq x$, then $y = x$.

[<u>Note</u>: Same as for maximal.]

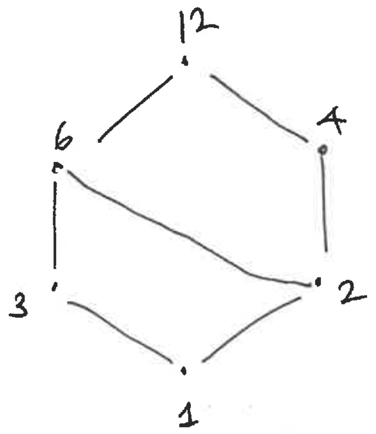
<u>Examples to play with</u>



* Algebra ~~of~~ on posets

Example $S = \{ \text{factors of } 12 \}$

Relation = "divisibility"



Def: Let $(P, \preceq)$ be a poset.

~~A Let $x \preceq y$~~ Let $x, y \in P$.

The (closed) interval $[x, y]$ is the set

$\{ z \in P \mid x \preceq z \text{ and } z \preceq y \}$.

E.g. In the poset above,

- $[1, 6] = \{1, 2, 3, 6\}$
- $[6, 1] = \emptyset$
- $[1, 2] = \{1, 2\}$
- $[3, 3] = \{3\}$