MATH 2301

22 Aug 2022

* Intervals on posets

Let $(P, \preceq)$ be a poset. Let $x, y \in P$

The interval $[x, y]$ is

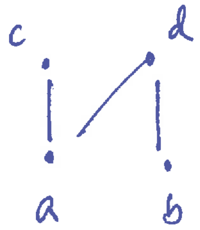
$$\{z \in P \mid x \preceq z \text{ and } z \preceq y\}$$

* The incidence algebra of a poset

Fix a poset $(P, \preceq)$.

The set $I(P)$ is the set of all non-empty intervals in P.

Example



$I(P) = \{[a,c], [a,d], [b,d], [a,a], [b,b], [c,c], [d,d]\}$

Def : The incidence algebra of $(P, \preceq)$ is the set of all functions

$$f : I(P) \to \mathbb{R},$$

together with two operations called $+$ and $*$ (to be defined soon.)

We'll denote the set $\{f : I(P) \to \mathbb{R}\}$ by $\mathcal{A}(P)$.

<u>Rmk</u>: To specify an element f of $\mathcal{A}(P)$, we need to say what value f takes on each non-empty interval in P.

<u>Examples</u>

1. (Previous example)
"the zero function", which sends any $[x,y] \mapsto 0$.

2. or, we could have the following function:

$$
\begin{aligned}
[a,c] &\mapsto 2/3 \\
[a,d] &\mapsto -\pi \\
[b,d] &\mapsto 0 \\
[a,a] &\mapsto 5 \\
[b,b] &\mapsto 2022 \\
[c,c] &\mapsto -7.375 \\
[d,d] &\mapsto 21
\end{aligned}
$$

[There are many, in fact infinitely many, options!]

3. $P =$ 

$I(P) = \{[a,a], [a,b], [b,b]\}$

$f \in \mathcal{A}(P)$ is specified by 3 real numbers

Some specially-named elements of $\mathcal{A}(P)$

Let $(P, \leq)$ be any poset.

1. The function that sends any element of $I(P)$, i.e. any non-empty interval, to 1.

   This element of $\mathcal{A}(P)$, i.e. this function, is called "ζ" (zeta).

   In other words, $\zeta \in \mathcal{A}(P)$:

   $$\zeta([x,y]) = 1 \quad \text{for every } [x,y] \in I(P).$$

2. We also set $\delta \in \mathcal{A}(P)$ to be the function:

   $$\delta([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}, \quad \text{for any } [x,y] \in I(P).$$

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* What can we do with $\mathcal{A}(P)$?

** Addition $(+)$

If $f \in \mathcal{A}(P)$ and $g \in \mathcal{A}(P)$, we define $(f+g) \in \mathcal{A}(P)$ as follows:

$$(f+g)([x,y]) := f([x,y]) + g([x,y]).$$

** Scalar multiplication $(\cdot)$

If $f \in \mathcal{A}(P)$, and $r \in \mathbb{R}$, we define

$(r \cdot f) \in \mathcal{A}(P)$ as follows:

$$(r \cdot f)([x,y]) = \underbrace{r}_{\text{real number}} \cdot \underbrace{f([x,y])}_{\text{real number}}$$

Examples

$P =$ 

① $(\zeta + \delta)([a,b]) = \zeta([a,b]) + \delta([a,b])$

$= 1 + 0 = 1$

② $(\zeta + \delta)([b,b]) = 2$

③ $(15\,\zeta)([x,y]) = 15$

④ $(15\,\delta)([x,y]) = \begin{cases} 15 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

What about multiplying $f \in \mathcal{A}(P)$ and $g \in \mathcal{A}(P)$?

** Pointwise product ($\cdot$)

If $f \in \mathcal{A}(P)$, $g \in \mathcal{A}(P)$, we set

$(f \cdot g) \in \mathcal{A}(P)$ as:

$$(f \cdot g)([x,y]) := \underbrace{f([x,y])}_{\in \mathbb{R}} \cdot \underbrace{g([x,y])}_{\in \mathbb{R}}$$

Warning: This is not the standard multiplication we'll use for $\mathcal{A}(P)$

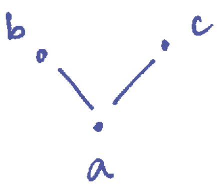
Instead, we'll use the "convolution product".

** Convolution product ($*$)

If $f, g \in \mathcal{A}(P)$, we define $(f * g) \in \mathcal{A}(P)$ as follows:

$$(f * g)([x,y]) = \sum_{x \leq z \leq y} f([x,z]) \cdot g([z,y])$$

dummy variable

Example:  , $(3 * 3)$

- $$(\zeta * \zeta)([a,a]) = \sum_{a \preceq z \preceq a} \zeta([a,z]) \cdot \zeta([z,a])$$

$$= \zeta([a,a]) \cdot \zeta([a,a]) = 1$$

($z = a$ only possible value)

- $$(\zeta * \zeta)([a,b]) = \sum_{a \preceq z \preceq b} \zeta([a,z]) \cdot \zeta([z,b])$$

$$= \zeta([a,a])\,\zeta([a,b]) + \zeta([a,b])\,\zeta([b,b])$$

$$= 2 .$$