

* Mid-semester in 1 week!

* Convolution product on $\mathcal{A}(P)$
↑ incidence algebra of poset P

$$\mathcal{A}(P) = \{f : I(P) \rightarrow \mathbb{R}\}$$

↑ non-empty intervals of P

$$(f * g)([x, y]) = \sum_{x \leq z \leq y} f([x, z]) \cdot g([z, y])$$

Note: $f * g$ may not be equal to $g * f$!

We had $\zeta, \delta \in \mathcal{A}(P)$

Calculation

Let $f \in \mathcal{A}(P)$; let us compute $f * \delta$.

$$(f * \delta)([x, y]) = \sum_{x \leq z \leq y} f([x, z]) \cdot \delta([z, y])$$

= 0 unless $z = y$

$$(f * \delta)([x, y]) = f([x, y]) \cdot \delta([y, y]) = \underline{f([x, y])}$$

$$\Rightarrow f * \delta = f$$

$$(\delta * f)([x, y]) = \sum_{x \leq z \leq y} \underbrace{\delta([x, z])}_{= 0 \text{ unless } x=z} \cdot f([z, y])$$

$$(\delta * f)([x, y]) = \delta([x, x]) \cdot f([x, y]) = f([x, y])$$

Prop: $\delta \in \mathcal{A}(\mathbb{P})$ is the multiplicative identity for convolution, i.e.: for every $f \in \mathcal{A}(\mathbb{P})$,

$$\delta * f = f * \delta = f.$$

Rmk: Note that $*$ is an associative operation;

$$\text{i.e. } (f * g) * h = f * (g * h)$$

Def: We say that an element $f \in \mathcal{A}(\mathbb{P})$ is invertible (with respect to $*$) if there is some

$g \in \mathcal{A}(\mathbb{P})$, such that:

$$f * g = \delta \quad (\text{equivalently, if } g * f = \delta)$$

Question: Is δ invertible? Yes: $\delta * \delta = \delta$.

Question: Is \mathbb{Z} invertible?

[Answer this soon...]

Example



$$f([a, a]) = 1$$

$$f([b, b]) = 0$$

$$f([c, c]) = 1$$

$$f([a, b]) = 2$$

$$f([a, c]) = 5$$

, an element of $\mathcal{A}(P)$.

Q: Is this invertible? No

If we had an inverse g , we'd have

$$(f * g)([b, b]) = \delta([b, b]) = 1$$

"

$$\sum_{b \leq z \leq b} f([b, z]) g([z, b])$$

$$f([b, b]) \cdot g([b, b])$$

$$= 0 \cdot g([b, b])$$



How about $f([a, a]) = f([b, b]) = f([c, c]) = 1$
 $f([a, b]) = 2, f([a, c]) = 5$?



Let's try and find the inverse ...

Suppose $g \in \mathcal{A}(P)$ so that $f * g = \delta$.

$$\Rightarrow (f * g)([x, y]) = \delta([x, y])$$

$$\text{So, } (f * g)([a, a]) = (f * g)([b, b]) = (f * g)([c, c]) = 1$$

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$$\sum_{a \leq z \leq a} f([a, z]) g([z, a]) =$$

$$f([a, a]) \cdot g([a, a]) = 1$$

i.e. $g([a, a]) = 1$ (similarly for $[b, b]$ & $[c, c]$)

$$(f * g)([a, b]) = \sum_{a \leq z \leq b} f([a, z]) \cdot g([z, b])$$

$$\begin{aligned} & \parallel \\ & \delta([a, b]) \\ & \parallel \\ & 0 \end{aligned}$$

$$= \underbrace{f([a, a])}_{\parallel 1} \cdot g([a, b]) + \underbrace{f([a, b])}_{\parallel 2} \cdot \underbrace{g([b, b])}_{\parallel 1}$$

$$\Rightarrow g([a, b]) = -2$$

$$\text{Similarly, } g([a, c]) = -5$$

\Rightarrow We found a g that works!

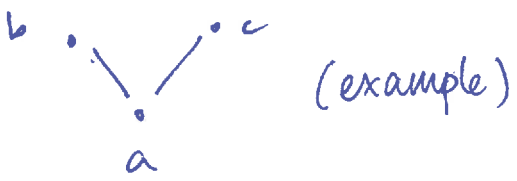
Rmk

(1) $f * g = \delta$ if and only if $g * f = \delta$

(2) If f is invertible, then it has a unique inverse.

(3) If f is invertible, we call its inverse f^{-1}
Not the usual inverse function!!

Question: Is ζ invertible?



If ζ^{-1} exists, then $(\zeta * \zeta^{-1})([x, y]) = \delta([x, y])$

As in the previous calculation, $\zeta^{-1}([x, x]) = 1 \neq x$.

$$(\zeta * \zeta^{-1})([a, b]) = \sum_{a \leq z \leq b} \underbrace{\zeta([a, z])}_{=1} \cdot \zeta^{-1}([z, b])$$

$$\underbrace{\delta([a, b])}_{=0} = \zeta^{-1}([a, b]) + \underbrace{\zeta^{-1}([b, b])}_{=1}$$

$$\Rightarrow \zeta^{-1}([a, b]) = -1$$

$$\zeta^{-1}([a, c]) = -1 \text{ (similarly)}$$

* Prop: ζ is always invertible in a finite poset,
and ζ^{-1} is called μ (mu), or the Möbius function