MATH 2301

24 Aug 2022

* Mid-sem in 1 week!

* Convolution product on $\mathcal{A}(P)$

$\uparrow$ incidence algebra of poset P

$$\mathcal{A}(P) = \{ f : I(P) \to \mathbb{R} \}$$

$\uparrow$ non-empty intervals of P

$$(f*g)([x,y]) = \sum_{x \preceq z \preceq y} f([x,z]) \cdot g([z,y])$$

Note: $f*g$ may not be equal to $g*f$!

We had ζ, $\delta \in \mathcal{A}(P)$

Calculation

Let $f \in \mathcal{A}(P)$; let us compute $f*\delta$.

$$(f*\delta)([x,y]) = \sum_{x \preceq z \preceq y} f([x,z]) \cdot \underbrace{\delta([z,y])}_{\substack{=0 \text{ unless} \\ z=y}}$$

$$(f*\delta)([x,y]) = f([x,y]) \cdot \delta([y,y]) = f([x,y]).$$

$\Rightarrow$ $f*\delta = f$

$$(\delta * f)([x,y]) = \sum_{x \leq z \leq y} \underbrace{\delta([x,z])}_{\substack{=0 \\ \text{unless } x=z}} \cdot f([z,y])$$

$$(\delta * f)([x,y]) = \delta([x,x]) \cdot f([x,y]) = f([x,y])$$

Prop: $\delta \in \mathcal{A}(P)$ is the multiplicative identity for convolution, i.e: for every $f \in \mathcal{A}(P)$,

$$\delta * f = f * \delta = f.$$

Rmk: Note that $*$ is an associative operation; i.e

$$(f*g)*h = f*(g*h)$$

Def: We say that an element $f \in \mathcal{A}(P)$ is invertible (with respect to $*$) if there is some $g \in \mathcal{A}(P)$, such that:

$$f*g = \delta \quad (\text{equivalently, if } g*f = \delta)$$

Question: Is δ invertible? Yes: $\delta * \delta = \delta$.

Question: Is ζ invertible?

[Answer this soon...]

Example

$$f([a,a]) = 1$$
$$f([b,b]) = 0$$
$$f([c,c]) = 1 \quad \text{, an element of } \mathcal{A}(P).$$
$$f([a,b]) = 2$$
$$f([a,c]) = 5$$

Q: Is this invertible? <u>No</u>

If we had an inverse g, we'd have

$$(f*g)([b,b]) = \delta([b,b]) = 1$$
$$\| $$
$$\sum_{b \le z \le b} f([b,z])\, g([z,b])$$
$$\|$$
$$f([b,b]) \cdot g([b,b])$$
$$= 0 \cdot g([b,b])$$

!

How about $f([a,a]) = f([b,b]) = f([c,c]) = 1$
$f([a,b]) = 2, \quad f([a,c]) = 5$?

Let's try and find the inverse ...

Suppose $g \in \mathcal{A}(P)$ so that $f * g = \delta$.

$$\Rightarrow \quad (f*g)([x,y]) = \delta([x,y])$$

So, $(f*g)([a,a]) = (f*g)([b,b]) = (f*g)([c,c]) = 1$

$$(f*g)([a,a]) = \sum_{a \leq z \leq a} f([a,z])\, g([z,a]) = f([a,a]) \cdot g([a,a]) = 1$$

i.e. $g([a,a]) = 1$ (similarly for $[b,b]$ & $[c,c]$)

$$0 = \delta([a,b]) = (f*g)([a,b]) = \sum_{a \leq z \leq b} f([a,z]) \cdot g([z,b])$$

$$= \underbrace{f([a,a])}_{=1} \cdot g([a,b]) + \underbrace{f([a,b])}_{=2} \cdot \underbrace{g([b,b])}_{=1}$$

$$\Rightarrow \quad g([a,b]) = -2.$$

Similarly, $g([a,c]) = -5$.

$\Rightarrow$ We found a g that works!

Rmk

(1) $f * g = \delta$ if and only if $g * f = \delta$

(2) If f is invertible, then it has a unique inverse.

(3) If f is invertible, we call its inverse f^{-1}

Not the usual inverse function!!

Question: Is ζ invertible?

(example)

If ζ^{-1} exists, then $(\zeta * \zeta^{-1})([x,y]) = \delta([x,y])$

As in the previous calculation, $\zeta^{-1}([x,x]) = 1 \quad \forall x$.

$$\underbrace{(\zeta * \zeta^{-1})([a,b])}_{\delta([a,b]) = 0} = \sum_{a \leq z \leq b} \underbrace{\zeta([a,z])}_{=1} \cdot \zeta^{-1}([z,b])$$

$$= \zeta^{-1}([a,b]) + \underbrace{\zeta^{-1}([b,b])}_{=1}$$

$\Rightarrow \quad \zeta^{-1}([a,b]) = -1$

$\zeta^{-1}([a,c]) = -1$ (similarly)

$*$ **Prop**: ζ is always invertible in a finite poset, and ζ^{-1} is called μ (mu), or the Möbius function