MATH 2301 — 25 Aug 2022

* Midterm $\equiv$ wednesday!
* Extra office hours: 1:30 – 3:00 pm Friday 26^{th} (Zoom)
  2:00 – 3:00 pm Monday 29^{th} (Hybrid)
  Usual on Wed 31^{st} & 3:00 – 4:00 (office)

---

* $(P, \leq)$ a poset

$\mathcal{A}(P)$ = incidence algebra $= \{ f : I(P) \to \mathbb{R} \}$

Convolution product

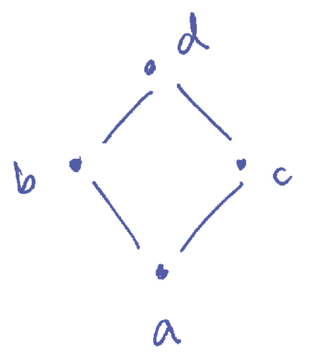
Inverses w.r.t convolution product:

$f, g \in \mathcal{A}(P)$ are inverses if $f * g = \delta$

Recall: $\zeta \in \mathcal{A}(P)$ is always invertible.

ζ^{-1} is known as mu $= \mu$ (Möbius function)

E.g.



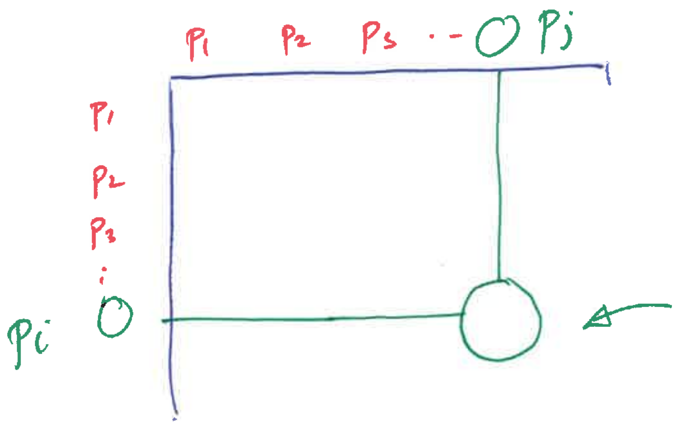
* Today: Matrix representations of $\mathcal{A}(P)$.

Idea: Represent elements of $\mathcal{A}(P)$ as matrices.

Let $f \in \mathcal{A}(P)$

(1) ~~Make a~~ Order the elements of $P = (p_1, \cdots, p_n)$ (preferably, in a topological sorting)

(2) Make an $n \times n$ matrix $[n = |P|]$, rows and cols indexed by elements of P



$f([p_i, p_j])$ if $[p_i, p_j]$ is non-empty.

0 if $[p_i, p_j] = \emptyset$

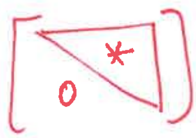
E.g. (previous example)

$f([x,y]) = \#$ elements in $[x,y]$, if $[x,y] \neq \emptyset$.

matrix:

$$M_f = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note: this is upper-triangular



* <u>Proposition</u> : If we choose a topological ordering for the elements of P, then the matrix of any $f \in \mathcal{A}(P)$ will be upper-triangular.

If $f \in \mathcal{A}(P)$, we call the corresponding matrix M_f.

** <u>Addition</u>

$f, g \in \mathcal{A}(P)$

$$(f+g)([x,y]) = f([x,y]) + g([x,y])$$

$$(M_f + M_g)_{(x,y)} = (M_f)_{(x,y)} + (M_g)_{(x,y)}$$

<u>Prop</u> : The matrix of $(f+g)$ is just $M_f + M_g$

i.e, $M_{(f+g)} = M_f + M_g$

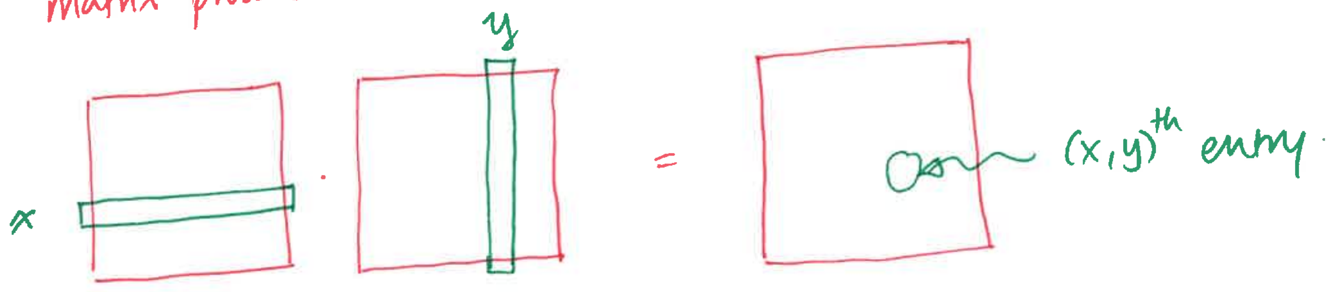
** <u>Convolution product</u>

$$(f * g)([x,y]) = \sum_{x \leq z \leq y} f([x,z]) \cdot g([z,y])$$

What about

$$(M_f \cdot M_g)_{(x,y)}$$

matrix product.



x^{th} row of M_f : $\left((M_f)_{(x,p_1)}, (M_f)_{(x,p_2)}, \dots, (M_f)_{(x,p_n)}\right)$

y^{th} column of M_g : $\begin{pmatrix} (M_g)_{(p_1,y)} \\ (M_g)_{(p_2,y)} \\ \vdots \\ (M_g)_{(p_n,y)} \end{pmatrix}$

$$(M_f \cdot M_g)_{(x,y)} = \sum_{p_i} \underbrace{(M_f)_{(x,p_i)}}_{\substack{=0 \text{ unless} \\ x \le p_i}} \cdot \underbrace{(M_g)_{(p_i,y)}}_{\substack{=0 \text{ unless} \\ p_i \le y}}$$

$$(M_f \cdot M_g)_{(x,y)} = \sum_{x \le z \le y} (M_f)_{(x,z)} \cdot (M_g)_{(z,y)}$$

** Proposition : The matrix of $(f*g)$ is just ~~Mf~~ the product $M_f \cdot M_g$, i.e

$$M_{(f*g)} = M_f \cdot M_g$$

convolution product

matrix product

** Proposition : ~~Then~~ Let $f \in A(P)$ be invertible. Then the matrix of f^{-1} is the inverse of the matrix M_f.

(More on Monday!)